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The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods.

mathematical models and methods

Unit 26
Vector Calculus

The Open University

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Unit 26

Vector calculus

Prepared for the Course Team
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The Open University

The Open University, Walton Hall, Milton Keynes.

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Introduction

In *Unit 14* on vector algebra we showed how a physical quantity could be categorized as either a scalar or a vector, and we spent some time on the algebra of vectors, i.e. adding, scaling and multiplying vectors. A *scalar* quantity has magnitude only and can be represented by a real number; for instance, mass and temperature are scalars. A *vector* quantity has two properties associated with it, its magnitude and a definite direction in space. We represent a vector on a diagram by an arrow or algebraically by components. Velocity and force are examples of physical quantities which can be represented by vectors.

In *Unit 14* the values of the scalars and the components of the vectors did not depend on where they were measured.

In this unit we shall consider scalars and vectors which have different values at different points in a region of space. For instance, you know from practical experience that the temperature in a bowl of hot soup varies quite significantly from the middle to the edge of the bowl. That is why we try to drink the soup from the edge rather than from the middle where we are likely to get a burnt mouth! Temperature is a scalar quantity and its value depends on where in the soup it is measured. The temperature distribution is an example of a *scalar field*.

Similarly, the force on a nail in the region surrounding a magnet will depend on how near to the magnet we bring the nail. The magnetic force field of the magnet has a definite magnitude and direction which varies with the position of the point at which it is measured. This is an example of a *vector field*.

This unit is about scalar and vector fields.

Study guide

This unit is divided into four sections and they should be studied in the order in which they appear. The audio-tape is part of Subsection 3.3 and the television programme should be viewed at the beginning of Section 4. If you have not had time to study the first three sections before watching the television programme then a quick skim through Sections 1 and 3 should be sufficient for you to follow and enjoy the programme; then return and study the unit in more detail.

In Section 1 we define scalar and vector fields and show how we can describe them geometrically. In Section 2 we define the spatial rate of change of a scalar field called the *gradient* of that scalar field. Section 3 introduces a new kind of integral of a vector field called *the line integral*. You should listen to the tape for this unit while studying this section; it takes you through several line integrals to help you gain confidence at evaluating them. We then, in Section 4, define a new vector field related to the spatial rate of change of the old vector field and called its *curl*. This definition uses the line integral introduced in Section 3. The television programme, which is associated with this section, shows you the physical significance of curl by considering the velocity vector fields of three fluid flows.

Finally, a note to those students who have previously studied *MS283*. You should be familiar with the material contained in this unit. You are recommended to work through the exercises and to revise those sections whose exercises you find difficult.

1 Scalar and vector fields

1.1 A scalar field defined

Figure 1 shows the temperature at different points of a flat plate when unevenly heated and cooled. The value of the temperature depends on where on the plate it is measured. If we define Cartesian coordinate axes Ox and Oy as shown, then the variable T representing the temperature will be a function of x and y; i.e. $T = T(x, y)$. This function of two variables is just an example of the concept of a function of more than one variable introduced in *Unit 25*.

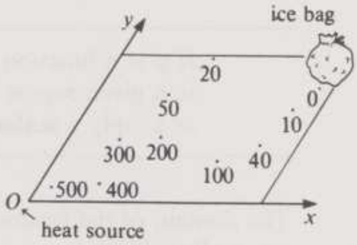


Figure 1

As another example, Figure 2 shows the (atmospheric) pressure distribution over the U.K. at midday on 30 September 1980.

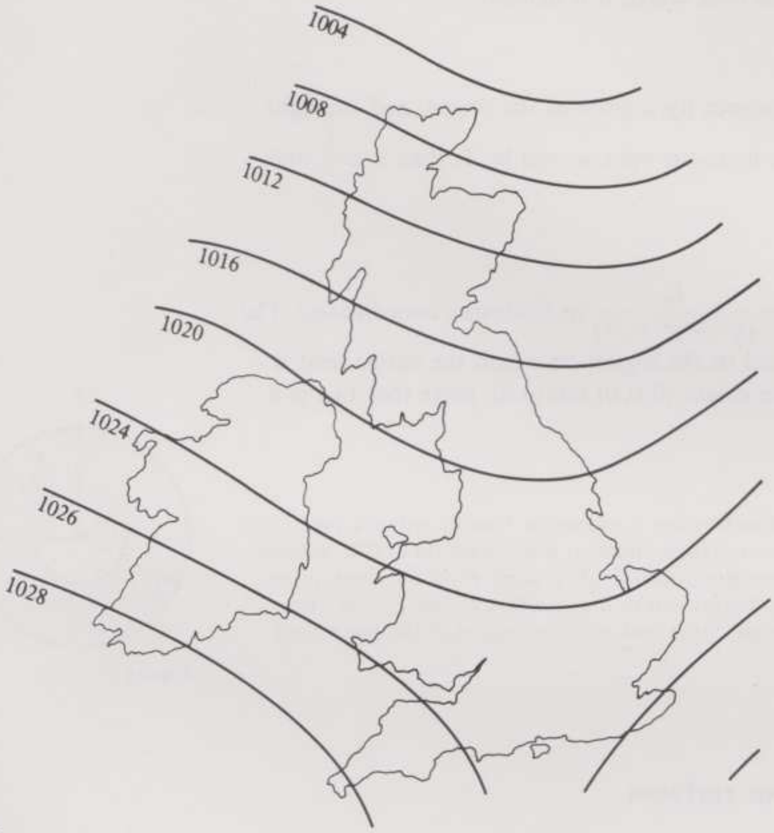


Figure 2

The solid lines are called *isobars* and join up places where the atmospheric pressure is the same. Atmospheric pressure at a given time is a scalar quantity and its value depends on where in the U.K. it is measured. For instance, at midday on 30 September 1980, in Manchester it was 1020 millibars and in Penzance it was 1028 millibars.

1 millibar = 100 N m^{-2}

If we choose London as an origin and set up a Cartesian coordinate system with Ox pointing East and Oy pointing North, then each place in the U.K. can be represented by its coordinates (x, y). Then the variable P representing the atmospheric pressure is a function of two variables x and y; i.e. $P = P(x, y)$.

These two examples are pictorial representations of *scalar fields*. In each of the examples above we have: (i) a scalar quantity (e.g. temperature or atmospheric pressure), (ii) a region of space over which each scalar is defined (e.g. the flat plate or the U.K.), and (iii) the values of the scalar quantity at different points of the region.

In each of the above cases, the variable representing the scalar quantity is a function of the two variables x and y; these are *two-dimensional* scalar fields. The temperature distribution in the room where you are reading this unit would be an example of a *three-dimensional* scalar field, which we now define:

> If ϕ is a function which associates a unique scalar with each point in a given region of space, then ϕ is called a **scalar field function** or simply a **scalar field**.

We can define a two-dimensional scalar field similarly, but then the domain is a region of the plane and not a region of space.

The domain of the function ϕ is the given region of space and the codomain is the set of all scalars.

Example 1

The light intensity from a point source of light, such as a small light bulb, is inversely proportional to the square of its distance from the bulb. The intensity at distance 1 metre is I_0. Specify a scalar field function which models the light intensity and state the region of space over which it is defined.

Solution

Suppose that we represent the light source by a point at the origin, and the light intensity at a point distance r metres from the light source by I; then $I \propto \dfrac{1}{r^2}$ and $I = I_0$ when $r = 1$. Thus $I = \dfrac{I_0}{r^2}$.

The scalar field function is $I = \dfrac{I_0}{r^2}$ (or $\dfrac{I_0}{(x^2 + y^2 + z^2)}$ in Cartesian coordinates). The function is not defined when $r = 0$ and so the region on which the scalar field is defined is the whole of space with the origin $(0, 0, 0)$ removed. Note that this is a three-dimensional scalar field.

Exercise 1

A thin circular sheet of metal of radius 4 metres has a concentric hole of radius 1 metre cut out of it (see Figure 3). The inner perimeter is maintained at a temperature of 0°C and the outer perimeter is maintained at a temperature of 100°C. At a point P on the sheet, given by position vector $\mathbf{r}$, the temperature is proportional to $\log_e r$, where $r = |\mathbf{r}|$. Specify the scalar field function which models the temperature and state the region of the plane over which it is defined.

[*Solution on p. 44*]

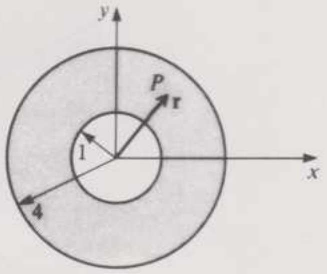


Figure 3

1.2 Contour curves and contour surfaces

The scalar fields in the last subsection were presented in three different ways. For the first scalar field of temperature over a plate, as given in Figure 1, we represented the data as values of the temperature at a few points on the plate. For the second scalar field of atmospheric pressure in the U.K., as given in Figure 2, we represented the data by isobars (or lines joining points at which the atmospheric pressure is the same) on a map of the U.K. (In both cases this is not a complete representation of the scalar field because for instance, there is a temperature associated with *every* point of the flat plate, and not just at the points shown in Figure 1.) Finally, in Example 1 and Exercise 1 the scalar fields were given by a formula of some type.

Of these modes of representation, the isobar example gives a very good way of visualizing pictorially the physical situation. Figure 4 shows the temperature distribution over an unevenly heated flat plate similar to that shown in Figure 1. However instead of giving the value of the temperature at different points as we did before, we have drawn curves through those points on the plate where the temperature is the same. Of course we cannot show all possible curves because that would require a curve through every point on the plate, so we choose certain values of the temperature and the curves give us a pictorial idea of the scalar field. So, for example, for each point on the curve AB the temperature equals 300°C.

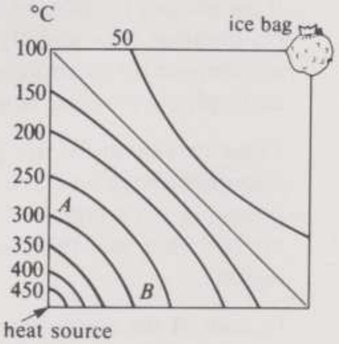


Figure 4

For a temperature field the curves are called **isotherms**, and in this example we have drawn the isotherms for multiples of 50°C. Notice that in one part of the plate the isotherms are much closer together than for the rest of the plate; the temperature is changing more rapidly in the region where the isotherms are close together.

We now extend this pictorial representation for pressure and temperature to more general two-dimensional scalar fields. Suppose we have a two-dimensional scalar field $\phi(x, y)$ defined over a region R. We can represent this scalar field pictorially by drawing curves through those points of R which have equal values of ϕ. Figure 5, for example, shows four such curves where C_1, C_2, C_3 and C_4 are constants. The curves that we have drawn are called **contour curves**.

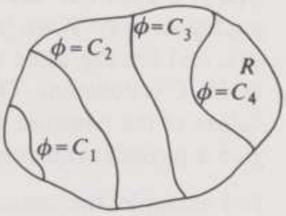


Figure 5

For example, consider the scalar field

$$\phi = x^2 + y^2$$

defined over the rectangle $-1 \leq x \leq 1$, $-2 \leq y \leq 2$. The contour curves are defined by

$$\phi = \text{constant}, \qquad \text{or} \qquad x^2 + y^2 = \text{constant}.$$

These curves are circles or parts of circles drawn within the rectangular region for which ϕ is defined. Figure 6 shows the contour curves

$$\phi(x, y) = 0.25; \qquad \phi(x, y) = 1; \qquad \phi(x, y) = 2.25; \qquad \phi(x, y) = 4.$$

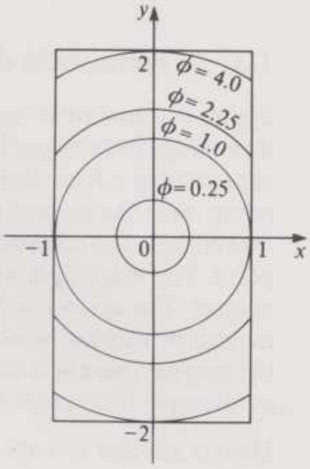


Figure 6

Again we cannot draw *all* possible contour curves, only some of them.

Exercise 2

Draw some of the contour curves to represent pictorially the scalar field $\phi = xy$ defined over the disc $x^2 + y^2 \leq 4$.

[*Solution on p. 44*]

You are probably familiar with the word 'contour' from Ordnance Survey maps. These are lines on a map joining up places which are at the same height above sea level. The contours can give us a lot of information about the lie of the land. Here is an example; see if you can answer Exercise 3, just from the contour curves.

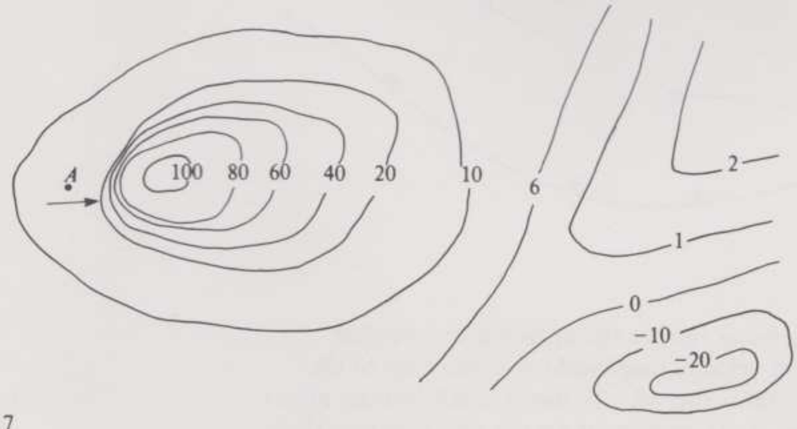


Figure 7

Exercise 3

(i) Where would you expect to find (a) a hill, and (b) a lake?

(ii) Where do you think that the land is fairly level?

(iii) Suppose you set off from point A and walked in the direction of the arrow, would you expect an easy walk or a hard climb?

[*Solution on p. 44*]

This exercise illustrates one important feature of a scalar field which can be obtained from its contour curves. The rate of increase (or decrease) of a scalar field can be deduced from the closeness of the contour curves. If the contour curves are close together then the scalar field is changing quite rapidly in that part of the region.

So far in this subsection we have considered two-dimensional fields. Now we turn our attention to three-dimensional scalar fields. A pictorial representation of a *two*-dimensional scalar field $\phi(x, y)$ is given by the contour *curves* defined by $\phi(x, y) = \text{constant}$. We can represent a three-dimensional scalar field $\phi(x, y, z)$ in an analogous way. If we join up points for which the three-dimensional scalar field $\phi(x, y, z)$ has the same value, we obtain the *surface* with equation $\phi(x, y, z) = C$ where C is constant. The family of surfaces given by $\phi(x, y, z) = C$, for different values of the constant C, are called the **contour surfaces** of the scalar field and they give a pictorial representation of the scalar field.

For example, the contour surfaces of the scalar field $I = I_0/(x^2 + y^2 + z^2)$ given in Example 1 are concentric spheres with centres at the origin. Two of the contour surfaces, for $I = 4$ and $I = 9$, are shown in Figure 8.

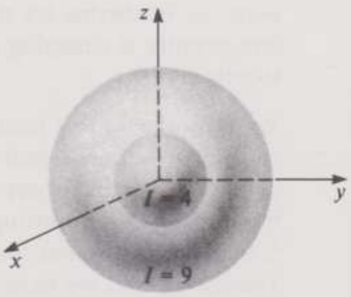


Figure 8

1.3 A vector field defined

Figure 9 shows what happens to iron filings when placed near a bar magnet. The iron filings have aligned themselves in a symmetrical pattern and they are experiencing a force due to magnetism. If we place small compasses at various points near the magnet we find the arrow on a compass orientates itself along the iron filings. The compasses show us the direction of the magnetic field at each point. The magnitude of the field decreases as we go further away from the magnet. The arrows in Figure 10 represent the magnetic field due to the bar magnet at various points near to the magnet. At each point in a region close to the magnet there is a unique vector called the **magnetic field** at that point. This is an example of a *vector field*.

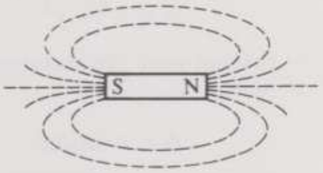


Figure 9

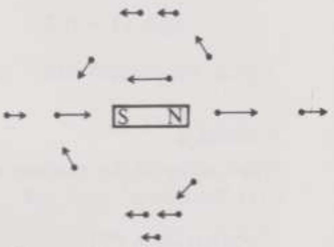


Figure 10

Here is another example. The arrows in Figure 11 represent the surface velocity at various points in a river.

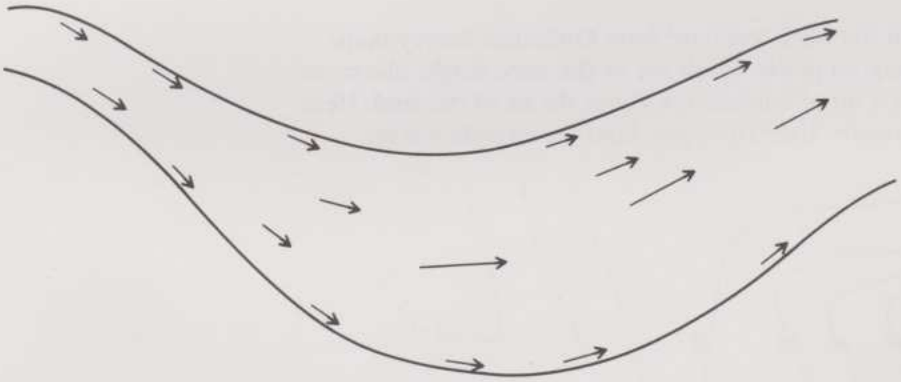


Figure 11

Near the river bank the water is almost at rest, so the arrows are somewhat shorter in length (indicating a smaller velocity magnitude near the edge of the river). The velocity at a particular point is the velocity that a small floating object, such as a leaf, would have at that point. At each point on the river's surface there is a unique velocity; this is an example of a **velocity vector field**.

You have seen something like this before in *Unit 2* where you met direction fields. These were a distribution of orientations in space, i.e. directions without a sense being attached, but no magnitudes were associated with the elements of the direction field. We now have a distribution of *vectors* in a region of space, i.e. quantities with *magnitude and direction*, which may take on different values at each point in the region. This leads to the following definition:

> If **F** is a function which associates a unique vector with every point in a given region of space, then **F** is called a **vector field function** or simply a **vector field**.

The domain of the function is the given region of space and the codomain is the set of all vectors. To describe a vector field completely we require the following: (i) the region of space on which the vector field is defined, and (ii) the magnitude and direction of the vector at each point of the region.

Example 2

The Earth's gravitational field can be considered as an example of a vector field: as you will see in *Unit 30*, the force on a body which is at a distance r from the centre of the Earth has magnitude inversely proportional to the square of the distance r and is directed towards the Earth's centre. Specify the vector field and the region of space over which it is defined.

Solution

The vector quantity is the gravitational force acting on the body due to the Earth. This force acts at points (x, y, z) in space such that

$$\sqrt{x^2 + y^2 + z^2} \geq R$$

where R is the radius of the Earth, and x, y, z are Cartesian coordinates with origin at the centre of the Earth. The vector field function is $-F_g\mathbf{e}_r$ where $\mathbf{e}_r$ is a unit vector whose direction is from O to P (see Figure 12). The magnitude F_g is proportional to $1/(OP)^2$, so

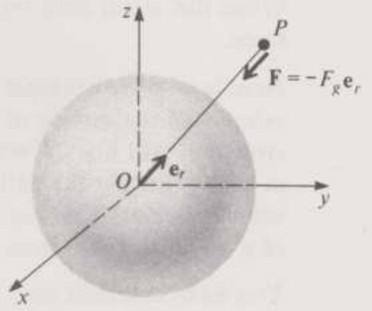


Figure 12

$$F_g = \frac{C}{OP^2} = \frac{C}{(x^2 + y^2 + z^2)}$$

for some constant C, and the direction of $\mathbf{F}$ is along the vector

$$\overrightarrow{PO} = -(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

so that

$$\mathbf{F} = -F_g\frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} = \frac{-C}{(x^2 + y^2 + z^2)}\frac{(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{-C(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2 + y^2 + z^2)^{3/2}}.$$

The region over which the vector field $\mathbf{F}$ is defined is the set of points (x, y, z) outside the sphere $x^2 + y^2 + z^2 = R^2$.

This is an example of a three-dimensional vector field because the function $\mathbf{F}$ is defined on a three-dimensional region of space.

The components of the vector field function relative to a given Cartesian coordinate system are themselves *scalar* field functions defined over the same region of space as $\mathbf{F}$. Thus if $\mathbf{F}$ is a three-dimensional vector field function defined on a region D and

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$$

then $F_1(x, y, z)$, $F_2(x, y, z)$ and $F_3(x, y, z)$ are scalar field functions also defined on D.

The following exercise gives an example of a two-dimensional vector field.

Exercise 4

A flat horizontal metal circular plate of radius 10 cm has a small hole in the middle through which is injected water at a constant rate. It is found that the velocity of the water at any point P on the plate is directed outwards along the line joining P to the centre of the plate O and has magnitude inversely proportional to the distance from O. Specify the vector field and the region over which it is defined, in terms of Cartesian coordinates with origin at O.

[*Solution on p. 44*]

1.4 Vector field lines

The arrows on the diagrams in the last subsection are one way of describing a vector field pictorially. The pictures are however incomplete because there is a vector quantity associated with *every* point in the region. A more complete picture can be obtained by drawing continuous curves in the region such that at any point the tangent to the curve is parallel to the direction of the vector field function at that point. These curves are called the **(vector) field lines** of a vector field. For instance, the orientation of iron filings near a bar magnet suggests a family of continuous curves. Two such curves are shown in Figure 13. At any point on one of the curves the orientation of a compass needle would be along the tangent to the curve; one is shown in the figure.

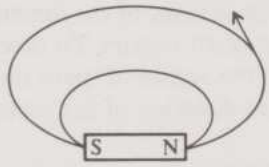


Figure 13

Exercise 5

Sketch the field lines for the Earth's gravitational force field.

[*Solution on p. 44*]

When the vector field represents a force the field lines are often called **lines of force**.

Here is another example of field lines. The arrows in Figure 14 represent the velocity of the surface of a river at various points. If we drop some leaves onto the river they will float. If we draw a line showing the path of one of these leaves, then at each point on the path, the tangent to the path is parallel to the direction of the velocity vector. This line is a field line for the velocity vector field. For the motion of a liquid the field lines are often called **flow lines**.

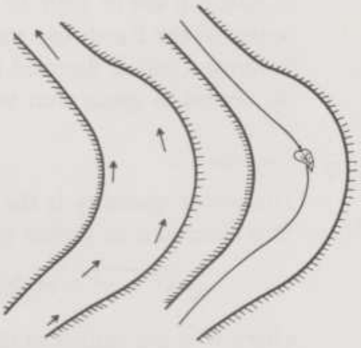


Figure 14

You have met field lines before in *Unit 2* of this course. The trajectories of a direction field are analogous to the field lines of a vector field.

The field lines give a convenient way of describing a vector field pictorially, and a diagram showing the field lines often helps us to describe the vector field which they illustrate. However the field lines tell us only the direction of the vector field and not the magnitude.

Example 3

Describe briefly the vector field corresponding to the field lines in Figure 15.

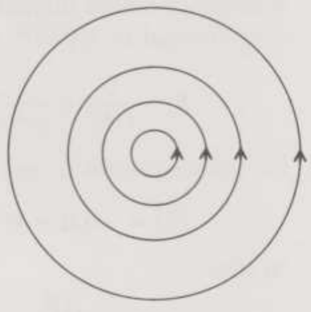


Figure 15

Solution

The vector at any point P is along the tangent to the circle through that point (see Figure 16), and this direction is perpendicular to the vector $\overrightarrow{OP}$. Since an anticlockwise sense around the circles is indicated by the arrowheads, the direction of $\mathbf{F}$ is that of $-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$. If P is the point (x, y) then $\sin\theta = y/\sqrt{x^2 + y^2}$ and $\cos\theta = x/\sqrt{x^2 + y^2}$, so the direction of $\mathbf{F}$ is that of $(-y\mathbf{i} + x\mathbf{j})$. The vector field function has the form $\mathbf{F} = f(x, y)(-y\mathbf{i} + x\mathbf{j})$ for some function $f(x, y)$ taking positive values only.

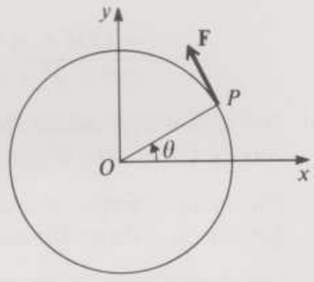


Figure 16

Exercise 6

Describe briefly the vector field corresponding to the field lines in Figure 17.

[*Solution on p. 44*]

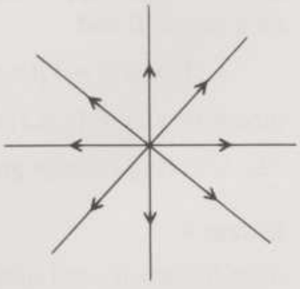


Figure 17

Summary of Section 1

1. If ϕ is a function which associates a unique scalar to each point in a given region of space, then ϕ is called a **scalar field function** or a **scalar field**.
2. A scalar field in *two* dimensions is defined over a region of the plane and can be represented by a function of two variables, $\phi(x, y)$. Such a scalar field can be described pictorially by **contour curves**; these are curves drawn on the x, y-plane with equations given by $\phi(x, y) = \text{constant}$.
3. A scalar field in *three* dimensions can be represented by a function of three variables, $\phi(x, y, z)$, and can be described pictorially by **contour surfaces**. These are surfaces with equations given by $\phi(x, y, z) = \text{constant}$.
4. If $\mathbf{F}$ is a function which associates a unique vector with every point in a given region of space, then $\mathbf{F}$ is called a **vector field function** or simply a **vector field**.

5. A vector field can be described pictorially by **field lines**. A field line is a curve whose tangent at each point is in the direction of the vector at the point.

2 Differentiating scalar fields

In the study of functions of more than one variable in the last unit, certain properties of the function were best understood in terms of the partial derivatives of the function. For instance, at a stationary point of a surface the first-order partial derivatives are zero. Whether the stationary point is a local maximum, local minimum or saddle point depends on some combination of the second-order partial derivatives.

Certain characteristics of scalar and vector fields can be described in terms of the *spatial* rate of change of the fields, and we shall see that these involve a combination of the first-order partial derivatives of the field functions. The properties that we are going to introduce in this and the next section provide us with relationships between a scalar field and a vector field, and between different vector fields. These are fundamental properties of fields that will be used in the last section and in *Unit 30* to characterize and describe the influences of various physical quantities in the real world.

By 'spatial' we mean with respect to space coordinates; that is, not with respect to time.

2.1 The gradient of a scalar field

A scalar field is described pictorially by contours, i.e. contour curves for a two-dimensional field and contour surfaces for a three-dimensional field. The closeness of the contours tells us quite a lot about how the field is changing in a particular direction. For instance, land elevations are represented by contour curves drawn on a map. If the contour curves are close together then there is a steep hill in that region and if the contours are well separated then the land is reasonably flat.

The change in a scalar field in different directions can also be described by looking at the contours. Suppose we consider the elevation of the land above sea level, represented by the function $\phi(x, y)$, as an example of a scalar field. Figure 1 shows a hill and its representation on a map by contour curves.

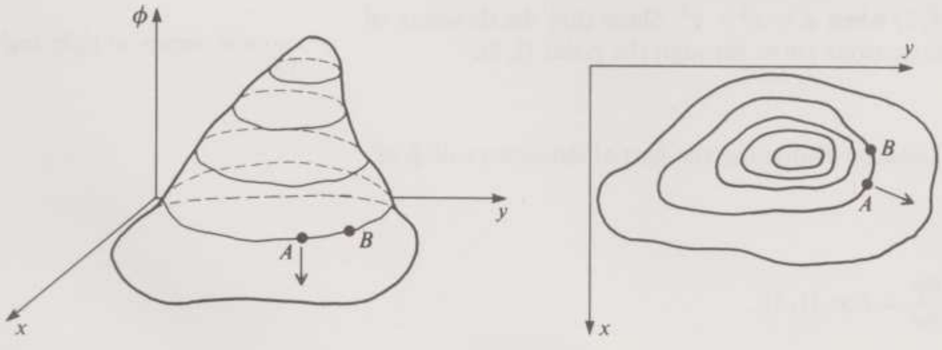


Figure 1

If we travel round a contour curve from A to B say, then the value of the function $\phi(x, y)$ does not change because a contour curve is defined by $\phi(x, y) = \text{constant}$. Now suppose that we put a ball at the point A on the hill. Experience tells us that the ball will begin to roll down the hill along the line of steepest slope, in other words the direction in which the scalar field is changing most rapidly. We would probably expect that this direction is perpendicular to the tangent to the curve AB at the point A. The acceleration of the ball will depend on how steep the slope of the hill is near the ball's position. At each point of the hill the magnitude and direction of the initial acceleration of a ball defines a vector field. We would expect this vector field to depend on how the scalar field of land elevations changes from point to point. It is this change in a scalar field that is described by the gradient.

In this subsection we are going to make this idea more precise. We begin by adopting the following definition of the gradient:

At each point in the domain of a two-dimensional scalar field ϕ we can evaluate the vector

$$\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j},$$

giving a vector field called the **gradient of the scalar field** ϕ and denoted by grad ϕ.

We will see that the gradient of a scalar field tells us how the scalar field is changing. But before we do that, the following examples and exercises show that finding the value of grad ϕ at a point is relatively straightforward.

Example 1

Evaluate grad ϕ at the point $(0, 1)$ when $\phi = \log_e(x + 2y)$.

Solution

The vector field grad ϕ is formed by evaluating the partial derivatives of ϕ of first order:

$$\phi = \log_e(x + 2y);$$

$$\frac{\partial\phi}{\partial x} = \frac{1}{x + 2y} \quad \text{and} \quad \frac{\partial\phi}{\partial y} = \frac{2}{x + 2y}.$$

So $\text{grad}\,\phi = \left(\frac{1}{x + 2y}\right)\mathbf{i} + \left(\frac{2}{x + 2y}\right)\mathbf{j}$.

Putting $x = 0$ and $y = 1$ we have, at the point $(0, 1)$, $\text{grad}\,\phi = \frac{1}{2}\mathbf{i} + \mathbf{j}$.

Exercise 1

Evaluate grad ϕ at the point $(-1, 2)$ when $\phi = x^2y$.

[*Solution on p. 44*]

Example 2

Evaluate grad ϕ at the point $(1, 2)$ when $\phi = x^2 + y^2$. Show that the direction of grad ϕ at $(1, 2)$ is normal to the contour curve through the point $(1, 2)$.

'normal' means 'at right angles to'

Solution

The vector grad ϕ at $(1, 2)$ is found by evaluating the partial derivatives of ϕ of first order at the point $(1, 2)$:

$$\phi = x^2 + y^2;$$

$$\frac{\partial\phi}{\partial x} = 2x \quad \text{and} \quad \frac{\partial\phi}{\partial x} = 2 \text{ at } (1, 2);$$

$$\frac{\partial\phi}{\partial y} = 2y \quad \text{and} \quad \frac{\partial\phi}{\partial y} = 4 \text{ at } (1, 2).$$

So at $(1, 2)$

$$\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} = 2\mathbf{i} + 4\mathbf{j}.$$

Let P be the point $(1, 2)$. The contour curve through P (see Figure 2) is a circle of radius $\sqrt{5}$. The normal to the circle through P is in the direction of the vector $\overrightarrow{OP} = \mathbf{i} + 2\mathbf{j} = \frac{1}{2}(2\mathbf{i} + 4\mathbf{j})$, so at $(1, 2)$ grad ϕ is in the same direction as this normal vector.

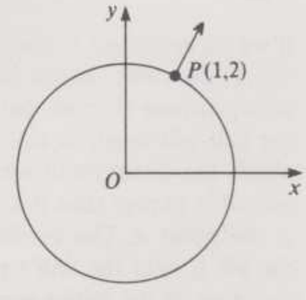


Figure 2

In fact we will now show that the direction of the gradient of *any* scalar field ϕ at a point is along the normal to the contour through that point.

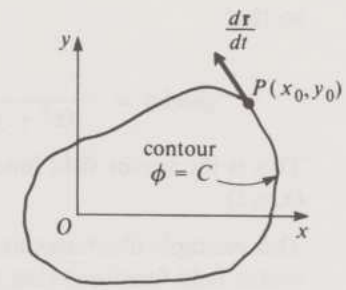


Figure 3

Suppose that $\phi(x, y)$ is a scalar field in two dimensions and consider one contour defined by $\phi(x, y) = C$ where C is some constant, and suppose that P is a point on the curve with coordinates (x_0, y_0) (see Figure 3).

In *Unit 14* we introduced the idea of a curve in the plane as the representation of some vector function of a scalar t; so the position of a point on the contour $\phi = C$ can be written as

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}$$

where t is a scalar, and as t varies the point moves along the contour. A tangent vector to the contour curve is then

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}.$$

With this representation the value of the scalar field $\phi(x, y)$ as we travel along the contour curve is also a function of t. Using the chain rule of *Unit 25* we can write

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt}$$

which is the dot product of two vectors, i.e.

$$\frac{d\phi}{dt} = \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j}\right)\cdot\left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right) = \mathbf{g}\cdot\frac{d\mathbf{r}}{dt}$$

where

$$\mathbf{g} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} = \operatorname{grad}\phi.$$

Since ϕ has constant value along the contour, $\frac{d\phi}{dt} = 0$; hence $\mathbf{g}\cdot\frac{d\mathbf{r}}{dt} = 0$, and the two vectors $\mathbf{g}$ and $\frac{d\mathbf{r}}{dt}$ are perpendicular. Thus the direction of grad ϕ is perpendicular to the tangent vector, i.e. grad ϕ *is along the normal to the contour.*

Exercise 2

Find a vector whose direction is normal to the curve $x^2 - 2xy + y^2 = 9$ at the point $(0, 3)$.

[*Solution on p. 44*]

For a three-dimensional scalar field represented by the field function $\phi(x, y, z)$, the gradient of ϕ is defined by

$$\boxed{\operatorname{grad}\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}.}$$

Example 3

Consider the scalar field $\quad \phi(x, y, z) = \dfrac{C}{(x^2 + y^2 + z^2)^{1/2}}$

where C is constant. Show that the gravitational field $\mathbf{F} = -F_g\mathbf{e}_r$ defined in Example 2 of Section 1 can be expressed in terms of this scalar field ϕ by the relation $\mathbf{F} = \operatorname{grad}\phi$.

Solution

For $\quad \phi = \dfrac{C}{(x^2 + y^2 + z^2)^{1/2}} \quad$ we have $\quad \dfrac{\partial\phi}{\partial x} = -\dfrac{Cx}{(x^2 + y^2 + z^2)^{3/2}},$

$$\frac{\partial\phi}{\partial y} = -\frac{Cy}{(x^2 + y^2 + z^2)^{3/2}} \quad \text{and} \quad \frac{\partial\phi}{\partial z} = -\frac{Cz}{(x^2 + y^2 + z^2)^{3/2}},$$

so that

$$\operatorname{grad}\phi = -\frac{C}{(x^2+y^2+z^2)^{3/2}}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}).$$

This is the vector field function describing the gravitational field at the point (x, y, z).

This example illustrates an important application of the gradient. Scalar and vector field functions that we use to model physical quantities in the real world are often related in this way; i.e. some vector field function is just a scalar multiple of the gradient of some scalar field function.

For instance, the gradient of the temperature field on the unevenly heated plate in Section 1 is a vector field which determines the flow of heat across the plate. In Figure 4 the dotted lines represent the contour curves and the arrows represent the gradient vector at various points on the plate.

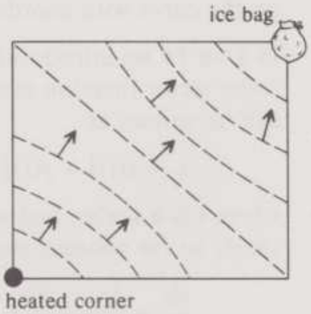


Figure 4

Exercise 3

Evaluate $\operatorname{grad}\phi$ at the point $(-1, 1, 0)$ for $\phi = x^2y + z^2x$.

[*Solution on p. 44*]

At a point P on a contour surface describing a three-dimensional scalar field, the gradient is perpendicular to the tangent plane at P.

Example 4

Find the direction of a normal vector to the surface $n_1x + n_2y + n_3z = C$ at the point $(1, 0, 2)$.

Solution

The surface $n_1x + n_2y + n_3z = C$ is a contour surface of the scalar field $\phi(x, y, z) = n_1x + n_2y + n_3z$ and the gradient of ϕ evaluated at the point $(1, 0, 2)$ is normal to the contour surface. We have

$$\frac{\partial\phi}{\partial x} = n_1, \qquad \frac{\partial\phi}{\partial y} = n_2 \qquad \text{and} \qquad \frac{\partial\phi}{\partial z} = n_3 \qquad \text{at } (1, 0, 2),$$

so that $\operatorname{grad}\phi = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$. This vector is the same for every point on the surface. This result should not surprise you because the equation $n_1x + n_2y + n_3z = C$ is the equation of a plane, as we saw in *Unit 14*.

The gradient of a scalar field ϕ is defined in terms of the partial derivatives of ϕ of first order and its direction is along the normal to the contour curves for a two-dimensional scalar field or along the normal to the tangent plane for a three-dimensional scalar field. So in some sense it is telling us about the change in ϕ as we go along the direction of the normal.

Consider again a two-dimensional scalar field ϕ; we now investigate the rate of change of ϕ in the direction of an arbitrary unit vector $\mathbf{e}$.

Figure 5 shows several contour curves for this scalar field and at a point P a unit vector $\mathbf{e}$. Suppose that the curve C is the path of a particle moving through the domain of the scalar field with unit speed, and that the direction of the tangent to the curve at the point P is given by the unit vector $\mathbf{e}$. Then the velocity of the particle as it passes through P is equal to $\mathbf{e}$. Using the idea of a curve as the representation of a vector function of time t as introduced in *Unit 14*, the position of the moving particle on the curve can be written as

$$\mathbf{r} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$$

and then $\mathbf{e} = \dfrac{d\mathbf{r}}{dt}$.

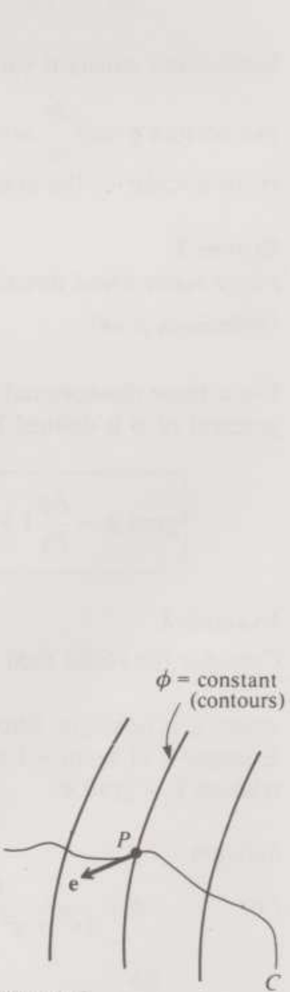


Figure 5

Using the chain rule of *Unit 25* we have the rate of change of ϕ with respect to t, at the point occupied by the moving particle, as

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} = (\text{grad}\,\phi)\cdot\frac{d\mathbf{r}}{dt}.$$

Now grad ϕ is along the normal to the contour at P, and $\dfrac{d\mathbf{r}}{dt}$ is the velocity of the moving particle which as we have seen is equal to **e**.

Suppose that θ is the angle between grad ϕ and **e** (see Figure 6). Then the above equation becomes

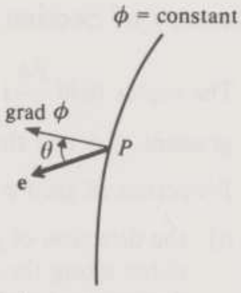


Figure 6

$$\frac{d\phi}{dt} = (\text{grad}\,\phi)\cdot\mathbf{e} = |\text{grad}\,\phi|\,|\mathbf{e}|\cos\theta = |\text{grad}\,\phi|\cos\theta$$

since **e** is a unit vector.

Considering how this expression varies with θ we see that the rate of change of ϕ is zero when $\theta = \pi/2$, i.e. along the contour. This is what we would expect because the contour is defined by $\phi =$ constant. The maximum rate of change in ϕ occurs when $\theta = 0$, i.e. along the direction of the vector grad ϕ, and this maximum rate of change of ϕ is $|\text{grad}\,\phi|$. This means that the gradient of a scalar field is the vector field which represents the *magnitude and direction of the maximum* (*spatial*) *rate of change of the scalar field*. This maximum rate of change is along the normal to (i) the contour curve for a two-dimensional scalar field and (ii) the contour surface for a three-dimensional scalar field.

Exercise 4

For $\phi = x^2y^2z^2$ find

(i) the maximum rate of change of ϕ at the point $(-1, 1, 1)$, and the direction in which this maximum occurs;

(ii) the rate of change of ϕ at the point $(2, 1, -1)$ in the direction of the unit vector $\mathbf{e} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$.

[*Solution on p. 45*]

2.2 The vector operator ∇

We now introduce a special notation for the gradient of a scalar field. Instead of grad ϕ we often write $\nabla\phi$ where the symbol ∇ is pronounced 'del' or 'nabla' and

$$\nabla \text{ stands for } \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

When written to the left of the symbol representing the scalar field function $\phi(x, y, z)$ it means 'find the first-order partial derivatives of ϕ and form the vector

$$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\text{'}.$$

The outcome of this operation is the gradient of the scalar field function. Usually we call $\nabla\phi$ not 'del ϕ' but 'grad ϕ'. ∇ is not a vector by itself but is called a **differential vector operator** implying that it needs something to 'operate on and differentiate' in a special way.

Exercise 5

We know that if f and g are two functions of a real variable t then

$$\frac{d}{dt}(fg) = \frac{df}{dt}g + f\frac{dg}{dt}.$$

What would be the analogous result for $\nabla(\phi\psi)$ where ϕ and ψ are two scalar fields?

[*Solution on p. 45*]

Summary of Section 2

1. The vector field $\dfrac{\partial \phi}{\partial x}\mathbf{i} + \dfrac{\partial \phi}{\partial y}\mathbf{j} + \dfrac{\partial \phi}{\partial z}\mathbf{k}$ formed from a scalar field ϕ is called the **gradient** of ϕ and this is written as grad ϕ or $\nabla\phi$.
2. Properties of grad ϕ:
   (i) the direction of grad ϕ at a point P in the domain of the scalar field ϕ, is either along the normal to the contour curve drawn through P for a two-dimensional scalar field or along the normal to the tangent plane of the contour surface drawn through P for a three-dimensional scalar field
   (ii) the magnitude of grad ϕ at a point P is the maximum rate of change of ϕ at the point P.
3. The rate of change of a scalar field ϕ at a point P in the direction of a unit vector $\mathbf{e}$ is given by $(\text{grad}\,\phi)\cdot\mathbf{e}$.

3 The scalar line integral

3.1 What is the scalar line integral?

You are familiar with the idea of integrating a continuous function over an interval $[a, b]$ of the real line, i.e. you can evaluate integrals of the form $\int_a^b f(x)\,dx$ which are called *definite integrals*. The definite integral is defined as the limit of a sum in the following way:

The definite integral is defined in *M101, Block III*; *MS283, Block III*; *TM281, Block IV*. See also the next unit in this course.

We divide the interval $[a, b]$ into N subintervals and in each subinterval we select a point:

x_1 in the first subinterval,

x_2 in the second subinterval,

$\vdots$

x_i in the ith subinterval

and so on.

Then we form the sum

$$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \cdots + f(x_i)\delta x_i + \cdots + f(x_N)\delta x_N$$

where δx_i is the width of the ith subinterval.

The limit of this sum as we increase the value of N indefinitely is defined to be the *definite integral of f from a to b* and is written

$$\int_a^b f(x)\,dx = \lim \sum_{i=1}^{N} f(x_i)\delta x_i.$$

This section is about generalizing the definite integral to give a rather special integral of a vector field function that occurs in applied mathematics; it is called the scalar line integral. You will see one application of it in Section 4 of this unit.

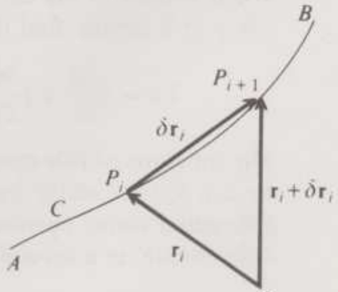


Figure 1

Consider a vector field $\mathbf{F}$ and let A and B be two points in the domain of $\mathbf{F}$ joined by a curve C (see Figure 1). Divide the curve into n segments and suppose P_i and P_{i+1} are two points on the curve with position vectors $\mathbf{r}_i$ and $\mathbf{r}_i + \delta\mathbf{r}_i$, where P_iP_{i+1} is one segment. At each point on the curve C the vector field has a particular magnitude and direction; at P_i let the vector field function be $\mathbf{F}_i$. The *scalar line integral of* $\mathbf{F}$ *along* C is defined to be $\lim \sum_{i=1}^{n} (\mathbf{F}_i \cdot \delta\mathbf{r}_i)$, and denoted by

$$\int_C \mathbf{F}\cdot d\mathbf{r}, \qquad \text{or sometimes by } \int_{AB} \mathbf{F}\cdot d\mathbf{r}.$$

If $\mathbf{F} = f(x)\mathbf{i}$ and C is the interval $[a, b]$ of the x-axis then $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b f(x)dx.$

The curve C is called the **path** of the integral. The path of a scalar line integral is said to be an **open path** if the two end points do not coincide, as in Figure 2, or a **closed path** if the two end points do coincide, as in Figure 3.

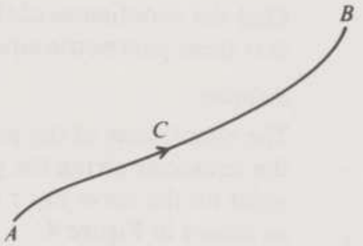


Figure 2: an open path

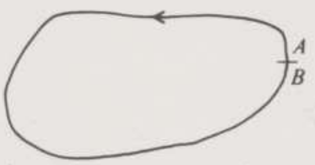


Figure 3: a closed path

We can integrate along the closed path in two directions, i.e. starting at A and ending at B or starting at B and ending at A. We usually adopt the convention of going *anticlockwise* round closed curves and to remind us of this we write the scalar line integral along a closed curve C as $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

The scalar line integral has many applications. For instance, if a vector field and a scalar field ϕ are such that $\mathbf{F} = \nabla\phi$, then given $\mathbf{F}$ we can find ϕ by evaluating the scalar line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$; we shall be more precise about this later in this section. In Section 4 we shall describe one of the properties of a vector field, called the *curl*, in terms of a scalar line integral.

Before we see how to evaluate scalar line integrals we introduce a method of representing every point on a path in terms of a single scalar *parameter*.

3.2 Parametrization of curves

In two dimensions a curve can often be described by an implicit equation $f(x, y) = 0$. For example, $f(x, y) = y - 2x - 3 = 0$ is the equation of a straight line or $f(x, y) = x^2 + y^2 - 9 = 0$ is the equation of a circle of radius 3 and centre the origin. In three dimensions this method is not suitable. An implicit equation $f(x, y, z) = 0$ describes a *surface*, not a curve.

We shall adopt a method to represent the paths of line integrals which applies to curves in two and three dimensions; the method is based on an idea introduced in *Unit 14*.

In *Unit 14* a vector function $\mathbf{f}$ of one real variable (t, say) was described pictorially by a *curve* in space so that any point on the curve had a position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{f}(t),$$

or in terms of the components,

$$\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k},$$

giving the coordinate equations

$$x = f_1(t), \qquad y = f_2(t), \qquad z = f_3(t).$$

These coordinate equations are called the **parametric equations** of the curve, and t is called the **parameter**; put in a value of t and we have the position vector of a point on the curve.

A parameter familiar to you from mechanics is *time*. As a particle moves in space subject to certain forces, its position vector is a function of time and the Cartesian coordinates of the particle's position at any time serve as the parametric equations of the path along which the particle moves. For instance, in a problem of a projectile moving in a vertical plane, the position vector of the particle at time t after projection is

$$\mathbf{r} = (u_0 t + x_0)\mathbf{i} + (-\tfrac{1}{2}gt^2 + v_0 t + z_0)\mathbf{k}$$

where $(x_0\,\mathbf{i} + z_0\,\mathbf{k})$ and $(u_0\,\mathbf{i} + v_0\,\mathbf{k})$ are the initial position and initial velocity of the particle respectively. The path described by the particle has parametric equations

$$x = u_0 t + x_0, \qquad y = 0, \qquad z = \tfrac{1}{2}gt^2 + v_0 t + z_0$$

and these equations represent a parabola in the x, z-plane. However, the parameter that we use in general need not have any physical significance.

Example 1

Consider the parametric equations

$$x = 2t, \qquad y = 4t^2 + 1, \qquad z = 2 \qquad (t \text{ any real number}).$$

Find the coordinates of the points for $t = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$, and sketch the curve that these parametric equations represent.

Solution

The coordinates of the points are found by substituting each value of t in turn into the equations giving the points $(-2,5,2)$, $(-1,2,2)$, $(0,1,2)$, $(1,2,2)$, $(2,5,2)$. Every point on the curve has $z = 2$ so that the curve lies completely in the plane $z = 2$, as shown in Figure 4.

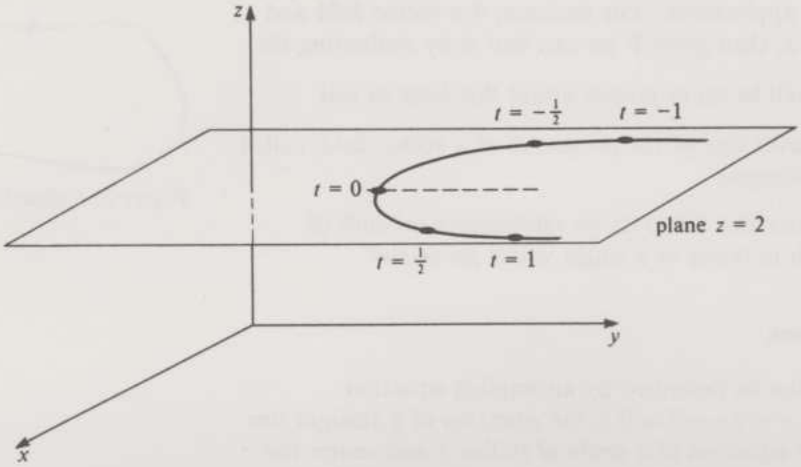


Figure 4

The parametric equations represent the parabola $y = x^2 + 1$ drawn in the plane $z = 2$.

Exercise 1

Consider the parametric equations

$$x = a\cos t, \qquad y = a\sin t, \qquad z = 0, \qquad (0 \le t < 2\pi).$$

Find the coordinates of the points for which $t = 0$, $t = \pi/4, \pi/2, 3\pi/4, \pi, 3\pi/2$, and sketch the curve that these parametric equations represent.

[*Solution on p. 45*]

The parametrization of a curve is not unique. For instance we can show that the points on a circle of radius a and centre the origin in the x,y-plane can be defined by

$$\text{(i) } x = a\cos t, \qquad y = a\sin t \qquad (0 \le t < 2\pi)$$

or

$$\text{(ii) } x = \frac{2at}{1+t^2}, \qquad y = \frac{a(1-t^2)}{1+t^2} \qquad (t \text{ any real number}).$$

In the above examples the parametrization of the curve has been given; but in two dimensions a path may be given as an implicit equation and finding a parametrization is not always easy. One method that will sometimes work is to let $x = t$ (or $y = t$) and then solve the implicit equation $f(t, y) = 0$ (or $f(x, t) = 0$) for y (or x) as a function of t.

For example, the general equation of a straight line is $y - ax - b = 0$ and if we let $x = t$ then $y = at + b$. These are the parametric equations of the straight line.

However, this method does not always give a parametrization of the entire curve. For instance, for the circle $x^2 + y^2 - a^2 = 0$, if we let $x = t$, (for $-a \leqslant t \leqslant a$) then $y^2 = a^2 - t^2$. Solving for y gives $y = \pm\sqrt{a^2 - t^2}$. The solution $y = +\sqrt{a^2 - t^2}$ gives a parametrization for the upper semicircle ($y > 0$) and $y = -\sqrt{a^2 - t^2}$ gives a parametrization for the lower semicircle ($y < 0$), but the method does not give a parametrization for the whole circle. The parametrization using the trigonometric functions, as in Exercise 1, does parametrize the whole circle.

Example 2

The general equation of an ellipse with centre the origin in the x,y-plane is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$. Find a parametrization for this curve.

Solution

We could try letting $x = at$ (for $-1 \leqslant t \leqslant 1$) so that $y^2 = b^2(1 - t^2)$, but then we would have to treat the two halves of the ellipse for $y > 0$ and $y < 0$ separately. For $y > 0$ we have $y = b\sqrt{1 - t^2}$ and for $y > 0$ we have $y = -b\sqrt{1 - t^2}$.

A parametrization which covers the whole curve is to let $x = a\cos t$ (for $0 \leqslant t < 2\pi$), and then $y = b\sin t$. (There are other parametrizations as well.)

What we are going to do is to use the parametrization of a curve to help us in the evaluation of a scalar line integral. Any parametrization will do, but for some the resulting integration is much easier than for others.

3.3 Evaluating the scalar line integral (Tape Subsection)

On the audio-tape that follows, we shall see how to evaluate the scalar line integral, $\int_C \mathbf{F}\cdot d\mathbf{r}$, for different vector fields $\mathbf{F}$ and different paths C. But first we shall show that by representing the path in terms of a parameter t, the scalar line integral can be written as a familiar looking integral in terms of t.

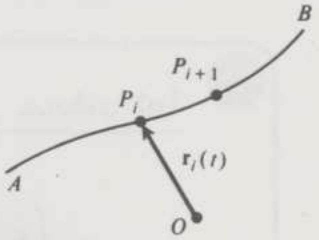


Figure 5

We defined the scalar line integral as the limit of a sum of the terms $\mathbf{F}_i\cdot\delta\mathbf{r}_i$ as the number of line segments gets larger and larger. The position vector $\mathbf{r}_i$ of the point P_i on the path C (see Figure 5) can be given in terms of a parameter t by

$$\mathbf{r}_i = \mathbf{r}(t_i) = x(t_i)\mathbf{i} + y(t_i)\mathbf{j} + z(t_i)\mathbf{k},$$

where t_i is the value of t corresponding to the point P_i.

As we move along the curve, t changes in value and the change in $\mathbf{r}_i$ for a small change in t is just

$$\delta\mathbf{r}_i = \frac{dx_i}{dt}\delta t\,\mathbf{i} + \frac{dy_i}{dt}\delta t\,\mathbf{j} + \frac{dz_i}{dt}\delta t\,\mathbf{k}$$

By $\frac{dx_i}{dt}$ we mean $\frac{dx}{dt}$ evaluated at t_i, that is, evaluated at the point P_i, and similarly for $\frac{dy_i}{dt}$ and $\frac{dz_i}{dt}$.

and we can write

$$\sum_{i=1}^{n} \mathbf{F}_i\cdot\delta\mathbf{r}_i = \sum_{i=1}^{n}\left(\mathbf{F}_i\cdot\frac{d\mathbf{r}_i}{dt}\right)\delta t.$$

In the limit as n increases indefinitely we can write

$$\lim\left(\sum_{i=1}^{n}\left(\mathbf{F}_i\cdot\frac{d\mathbf{r}_i}{dt}\right)\delta t\right) = \int_{t_1}^{t_2}\left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right)dt$$

where t_1 and t_2 are the values of t at the points A and B respectively.

The integral on the right hand side can be converted to a familiar looking integral: if we let $\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = f(t)$ then the scalar line integral becomes

$$\int_{t_1}^{t_2} f(t)\,dt.$$

The audio-tape develops an algorithm for evaluating scalar line integrals; it shows how to make the transformation from

$$\int_C \mathbf{F}\cdot d\mathbf{r} \quad \text{to} \quad \int_{t_1}^{t_2}\left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right)dt.$$

Start the tape when you are ready

① A two-dimensional line integral

Evaluate $\int_{C_1} \underline{F} \cdot d\underline{r}$

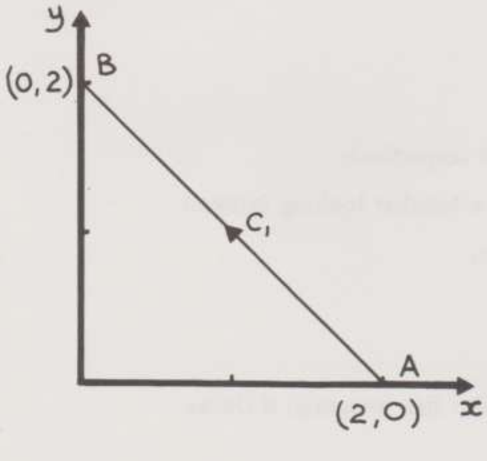
where $\underline{F} = (2x+y)\underline{i} - x\underline{j}$

and c_1 is the straight line between $(2,0)$ and $(0,2)$

② Introduce t

$$\int_{C_1} \underline{F} \cdot d\underline{r} = \int_{t_1}^{t_2} \left(\underline{F} \cdot \frac{d\underline{r}}{dt} \right) dt$$

③ The path



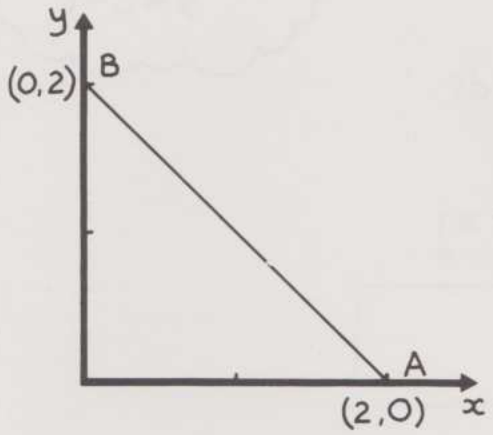
$y = ax + b$

a = gradient = ☐

b = y intercept = ☐

equation of line is $y =$

3a Answer



gradient $a = -1$

y intercept $b = 2$

equation $y = 2 - x$

4 Parametrize

$y = 2 - x$

$x = t$

$y = \boxed{}$

C_1 goes from $(2,0)$ to $(0,2)$

t goes from $\boxed{}$ to $\boxed{}$

⑤ Substitution

$x = t$

$y = 2 - t$

$$\int_{t_1}^{t_2}\left(\underline{F}\cdot\frac{d\underline{r}}{dt}\right)dt$$

$$\underline{F} = (2x + y)\,\underline{i} - x\underline{j}$$

$$\underline{F} = \boxed{\qquad\qquad}$$

$$\underline{r} = x\underline{i} + y\underline{j}$$

$$\underline{r} = \boxed{\quad}\,\underline{i} + \boxed{\quad}\,\underline{j}$$

$$\frac{d\underline{r}}{dt} = \boxed{\qquad\qquad}$$

$$\underline{F}\cdot\frac{d\underline{r}}{dt} = \boxed{\qquad\qquad}$$

Finally $t_1 = 2$ and $t_2 = 0$

Hence $\displaystyle\int_{t_1}^{t_2}\left(\underline{F}\cdot\frac{d\underline{r}}{dt}\right)dt = \int_{\boxed{\,}}^{\boxed{\,}}\boxed{\qquad\qquad}\,dt$

Solution at the end of the frames

6 Evaluation

$$\int_2^0 (2+2t)\,dt = \boxed{\qquad\qquad}$$

7 Line integrals

Step 1 : parametrize

look at path of integration

Step 2 : substitute

write all the variables x, y (etc) in terms of parameters

Step 3 : evaluate

this looks more like what I'm used to

8 Another 2-d problem

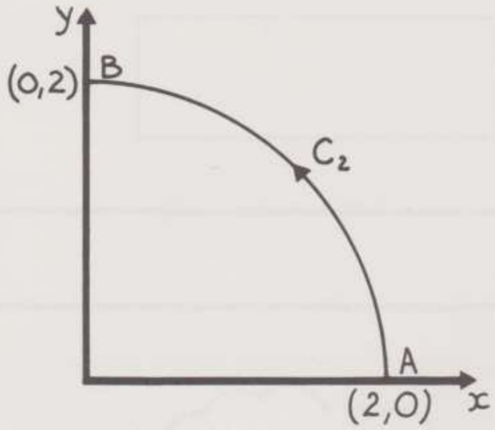
Evaluate $\displaystyle\int_{C_2} \underline{F} \cdot d\underline{r}$

where $\underline{F} = (2x+y)\,\underline{i} - x\underline{j}$

and C_2 is the quarter circle, centre at the origin and radius 2, between $(2,0)$ and $(0,2)$

9

Parametrize



Equation $x^2 + y^2 = 4$

$x = \boxed{\quad\quad}$

$y = \boxed{\quad\quad}$

C_2 starts at (2,0) and ends. at (0,2)

So t goes from

$t_1 = \boxed{\quad\quad}$ to $t_2 = \boxed{\quad\quad}$

10

Substitution

$x = 2\cos t$
$y = 2\sin t$

$$\int_{t_1}^{t_2} \left(\underline{F} \cdot \frac{d\underline{r}}{dt}\right) dt$$

$\underline{F} = (2x + y)\,\underline{i} - x\,\underline{j}$

$\underline{F} = \boxed{\quad\quad\quad\quad}$

$\underline{r} = x\,\underline{i} + y\,\underline{j}$

$\underline{r} = \boxed{\quad\quad}\,\underline{i} + \boxed{\quad\quad}\,\underline{j}$

$\frac{d\underline{r}}{dt} = \boxed{\quad\quad\quad\quad}$

$\underline{F} \cdot \frac{d\underline{r}}{dt} = \boxed{\quad\quad\quad\quad}$

$t_1 = 0$ and $t_2 = \pi/2$

Hence $\int_{t_1}^{t_2} \left(\underline{F} \cdot \frac{d\underline{r}}{dt}\right) dt = \int_{\boxed{\,}}^{\boxed{\,}} \boxed{\quad\quad\quad\quad}\, dt$

⓫

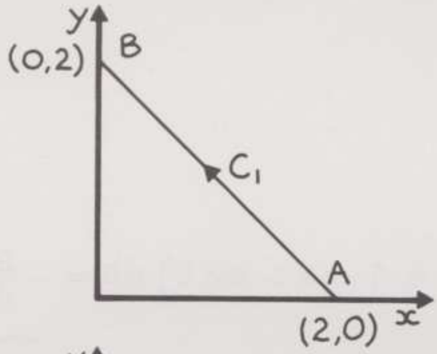
Evaluation

$$\int_0^{\pi/2} (-4 - 8 \sin t \cos t)\, dt$$

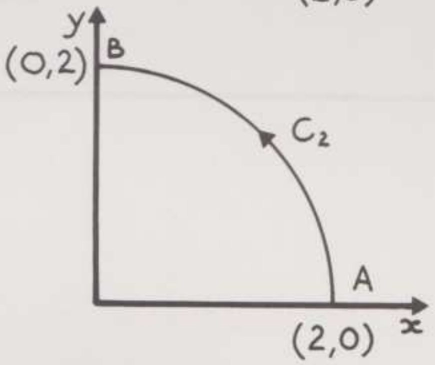
Solution at the end of the frames

⓬

Comparison of answers



$$\int_{C_1} \underline{F} \cdot d\underline{r} = -8$$



$$\int_{C_2} \underline{F} \cdot d\underline{r} = -2\pi - 4$$

⓭

Another vector field

evaluate $\int_C \underline{F} \cdot d\underline{r}$

where $\underline{F} = x^2\underline{i} + y\underline{j}$

(i) for $C = C_1$ (ii) for $C = C_2$

13a

Answer

(i) $C = C_1$

$$x = t \qquad y = 2 - t$$

$$\underline{F} = t^2\underline{i} + (2-t)\underline{j}$$

$$\frac{d\underline{r}}{dt} = \underline{i} - \underline{j}$$

$$\underset{\sim}{\int_{C_1} \underline{F} \cdot d\underline{r}} = \int_{t_1}^{t_2} \left(\underline{F} \cdot \frac{d\underline{r}}{dt}\right) dt = \int_2^0 (t^2 + t - 2)\, dt = \underset{\sim}{-\frac{2}{3}}$$

(ii) $C = C_2$

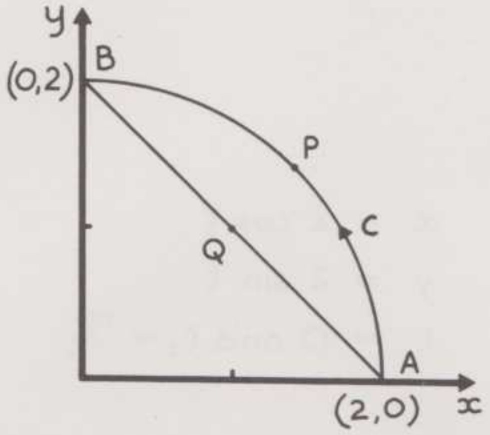
$$x = 2\cos t \qquad y = 2\sin t$$

$$\underline{F} = 4\cos^2 t\,\underline{i} + 2\sin t\,\underline{j}$$

$$\frac{d\underline{r}}{dt} = -2\sin t\,\underline{i} + 2\cos t\,\underline{j}$$

$$\underset{\sim}{\int_{C_2} \underline{F} \cdot d\underline{r}} = \int_{t_1}^{t_2} \left(\underline{F} \cdot \frac{d\underline{r}}{dt}\right) dt = \int_0^{\pi/2} (-8\cos^2 t\,\sin t + 4\cos t\,\sin t)\, dt = \underset{\sim}{-\frac{2}{3}}$$

⑭ A closed path



$$\oint_C \underline{F} \cdot d\underline{r} = ?$$

where $\underline{F} = xy\underline{i} + 2\underline{j}$

and C is the closed curve

APBQA

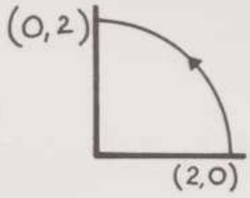
---

two
open
paths

$$\oint_C \underline{F} \cdot d\underline{r} = \int_{APB} \underline{F} \cdot d\underline{r} + \int_{BQA} \underline{F} \cdot d\underline{r}$$

(i) $\int_{APB} \underline{F} \cdot d\underline{r}$

Parametrize



substitute

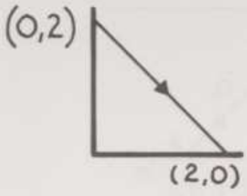
$$\int_{t_1}^{t_2} \left(\underline{F} \cdot \frac{d\underline{r}}{dt}\right) dt$$

evaluate

$$\int_{APB} \underline{F} \cdot d\underline{r} = \boxed{\quad}$$

(ii) $\int_{BQA} \underline{F} \cdot d\underline{r}$

Parametrize



substitute

$$\int_{t_1}^{t_2} \left(\underline{F} \cdot \frac{d\underline{r}}{dt}\right) dt$$

evaluate

$$\int_{BQA} \underline{F} \cdot d\underline{r} = \boxed{\quad}$$

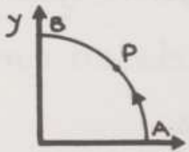
$$\oint_C \underline{F} \cdot d\underline{r} = \int_{APB} \underline{F} \cdot d\underline{r} + \int_{BQA} \underline{F} \cdot d\underline{r} = \boxed{\quad}$$

14a

Answers

(i) $\int_{APB} \underline{F} \cdot d\underline{r}$

Parametrize



$$x = 2\cos t$$
$$y = 2\sin t$$
$$t_1 = 0 \text{ and } t_2 = \pi/2$$

substitute

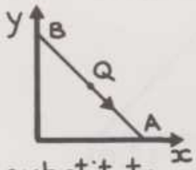
$$\underline{F} = 4\sin t\cos t\,\underline{i} + 2\,\underline{j}$$
$$\frac{d\underline{r}}{dt} = -2\sin t\,\underline{i} + 2\cos t\,\underline{j}$$

evaluate

$$\int_0^{\pi/2} (-8\sin^2 t\cos t + 4\cos t)\,dt = \frac{4}{3}$$

(ii) $\int_{BQA} \underline{F} \cdot d\underline{r}$

Parametrize



$$x = t$$
$$y = 2 - t$$
$$t_1 = 0 \text{ and } t_2 = 2$$

substitute

$$\underline{F} = t(2-t)\,\underline{i} + 2\underline{j}$$
$$\frac{d\underline{r}}{dt} = \underline{i} - \underline{j}$$

evaluate

$$\int_0^2 (2t - t^2 - 2)\,dt = -\frac{8}{3}$$

$$\oint_C \underline{F} \cdot d\underline{r} = \int_{APB} \underline{F} \cdot d\underline{r} + \int_{BQA} \underline{F} \cdot d\underline{r} = \frac{4}{3} - \frac{8}{3} = -\frac{4}{3}$$

⑮

What happens for $\underline{F} = x^2\underline{i} + y\underline{j}$?

$$\oint_C \underline{F}\cdot d\underline{r} = \int_{APB} \underline{F}\cdot d\underline{r} + \int_{BQA} \underline{F}\cdot d\underline{r} = \boxed{\qquad\qquad}$$

Solution at the end of the frames

⑯

Classification of vector fields

$\oint_C \underline{F}\cdot d\underline{r} = 0$ for every closed path CONSERVATIVE FIELD

$\oint_C \underline{F}\cdot d\underline{r} \neq 0$ for some closed path NON-CONSERVATIVE FIELD

⑰

Summary : How to evaluate line integrals

step 1 : parametrize

step 2 : substitute

step 3 : evaluate

This applies to three-dimensional line integrals as well as two-dimensional line integrals

Solutions to Exercises in the tape

Frame 5

$$\mathbf{F} = (2+t)\mathbf{i} - t\mathbf{j}$$

$$\mathbf{r} = t\mathbf{i} + (2-t)\mathbf{j}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j}$$

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = (2+t) + t = (2+2t)$$

$$t_1 = 2 \qquad \text{and} \qquad t_2 = 0$$

$$\int_2^0 (2+2t)\,dt$$

Frame 11

$$\int_0^{\pi/2} (-4 - 8\sin t\cos t)\,dt = \Big[-4t - 4\sin^2 t\Big]_0^{\pi/2}$$

$$= (-2\pi - 4) - (0)$$

$$= -4 - 2\pi.$$

Frame 15

The path APB is the same as the path C_2 and Frame 13a gives the solution to the line integral of $\mathbf{F} = x^2\mathbf{i} + y\mathbf{j}$ along C_2. We have

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{APB} \mathbf{F}\cdot d\mathbf{r} = -\tfrac{2}{3}.$$

The path BQA is the same as the path C_1 but here we go along the path from B to A. In Frame 13a we integrated from A to B and found that $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{AQB} \mathbf{F}\cdot d\mathbf{r} = -\tfrac{2}{3}$.

The integral from B to A is just minus this answer because we just reverse the limits.

Thus $\int_{BQA} \mathbf{F}\cdot d\mathbf{r} = \tfrac{2}{3}$.

3.4 Post-tape exercises

Exercise 2

Evaluate the scalar line integral $\oint_C \mathbf{u}\cdot d\mathbf{r}$ for each of the vector field functions $\mathbf{u}$ given below and where C is a circle defined by the parametric equations

$$x = a\cos t, \qquad y = a\sin t, \qquad z = 0 \qquad (0 \le t \le 2\pi).$$

(i) $\mathbf{u} = \dfrac{1}{x^2+y^2}(x\mathbf{i} + y\mathbf{j})$;

(ii) $\mathbf{u} = -y\mathbf{i} + x\mathbf{j}$.

[*Solution on p. 45*]

The solution to this exercise illustrates one important point that was introduced on the tape: that the value of a line integral round a closed curve may not be zero.

This exercise and the problems in the tape section are examples of line integrals along paths drawn in the x,y-plane — these we called *two-dimensional* line integrals. We shall now look at some paths in three dimensions. The same three-step approach that we introduced on the tape applies here; the steps are (i) parametrize, (ii) substitute and (iii) evaluate.

Example 3

Evaluate the scalar line integral of $\mathbf{F} = xy\mathbf{i} + yz^2\mathbf{j} + y^2z\mathbf{k}$ along the path C between the points $(0,0,0)$ and $(1,1,1)$ defined by the parametrization

$$x = t, \qquad y = t^2, \qquad z = t \qquad (0 \le t \le 1).$$

Solution

The parametrization of the path is given in the question; this is usually the case for curves in three dimensions — they are *defined* in terms of a parameter. The limits of the integration are $t_1 = 0$ and $t_2 = 1$.

Now we must replace x, y and z throughout in terms of t. We have

$$\mathbf{F} = t^3\,\mathbf{i} + t^4\mathbf{j} + t^5\,\mathbf{k}$$

and $$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = t\,\mathbf{i} + t^2\,\mathbf{j} + t\,\mathbf{k}$$

so that

$$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\,\mathbf{j} + \mathbf{k}$$

and $$\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = t^3 + 2t^5 + t^5 = t^3 + 3t^5.$$

On substitution the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ becomes

$$\int_0^1 (t^3 + 3t^5)dt.$$

This is a familiar looking integral and evaluating it in the usual way we get

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (t^3 + 3t^5)dt = \tfrac{3}{4}.$$

Exercise 3

Evaluate the scalar line integral of

$$\mathbf{F} = (2x)\mathbf{i} + (xz - 2)\mathbf{j} + xy\,\mathbf{k}$$

along the path C_1 between the points $(0, 0, 0)$ and $(1, 1, 1)$ defined by the parametrization

$$x = t, \qquad y = t^2, \qquad z = t^3 \qquad (0 \le t \le 1).$$

[*Solution on p. 45*]

Exercise 4

In Exercise 3 the path of the integration starts at $(0, 0, 0)$ and ends at $(1, 1, 1)$. Repeat the exercise for the path starting at $(1, 1, 1)$ and ending at $(0, 0, 0)$ and following the path of Exercise 3 in the opposite direction.

[*Solution on p. 45*]

The solution to Exercise 4 illustrates an important property of the scalar line integral. If we reverse the end points of the path then the value of the line integral changes sign, i.e. for the path C in Figure 6

$$\int_{AB} \mathbf{F}\cdot d\mathbf{r} = -\int_{BA} \mathbf{F}\cdot d\mathbf{r}.$$

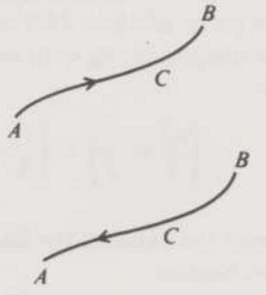


Figure 6

Exercise 5

(i) For the vector field defined in Exercise 3, evaluate $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ for the path C_2 between the points $(0, 0, 0)$ and $(1, 1, 1)$ defined by the parametric equations

$$x = t, \qquad y = t, \qquad z = t^2 \qquad (0 < t < 1).$$

(ii) Use the results of Exercise 3 and part (i) to find the value of $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where C is the closed curve starting at $(0, 0, 0)$ going along C_1 to $(1, 1, 1)$ and returning to $(0, 0, 0)$ along the path C_2.

[*Solution on p. 45*]

Example 4

Show that the scalar line integral $\int_{AB} \mathbf{F}\cdot d\mathbf{r}$ for the vector field $\mathbf{F} = x^2\,\mathbf{i} + y\,\mathbf{j}$ is independent of the path taken between A and B.

Solution

Consider two points A and B with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, and any curve C joining A to B (see Figure 7).

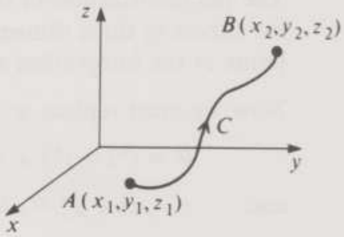


Figure 7

Let a parametrization of the curve C be

$$x = f(t), \qquad y = g(t), \qquad z = h(t),$$

with $t = t_1$ and $t = t_2$ at A and B respectively.

Then $\mathbf{F} = (f(t))^2\mathbf{i} + g(t)\mathbf{j}$,

and $\dfrac{d\mathbf{r}}{dt} = \dfrac{df}{dt}\mathbf{i} + \dfrac{dg}{dt}\mathbf{j} + \dfrac{dh}{dt}\mathbf{k}$,

so that
$$\int_{AB} \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t_1}^{t_2} \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right) dt$$
$$= \int_{t_1}^{t_2} \left(f^2\frac{df}{dt} + g\frac{dg}{dt}\right) dt$$
$$= \left[\frac{f^3}{3} + \frac{g^2}{2}\right]_{t_1}^{t_2}$$
$$= \left(\frac{(f(t_2))^3}{3} + \frac{(g(t_2))^2}{2}\right) - \left(\frac{(f(t_1))^3}{3} + \frac{(g(t_1))^2}{2}\right).$$

Now $f(t_1)$ and $f(t_2)$ are the x-coordinates of the points A and B; also $g(t_1)$ and $g(t_2)$ are the y-coordinates of the points A and B. The value of the line integral can thus be written in terms of the coordinates of A and B as follows:

$$\int_{AB} \mathbf{F}\cdot d\mathbf{r} = \left(\frac{x_2^3}{3} + \frac{y_2^2}{2}\right) - \left(\frac{x_1^3}{3} + \frac{y_1^2}{2}\right).$$

This result depends only on the positions of the end points A and B and *not* on the path joining A and B.

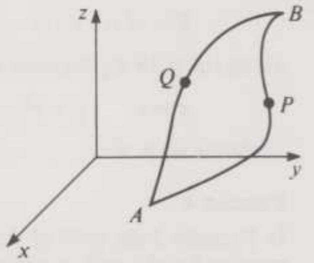


Figure 8

The vector field in Exercise 5 is an example of a **non-conservative** vector field because in part (ii) we found that the value of its line integral round a closed curve is non-zero. Compare this with the vector field defined in Example 4. Consider *any* two paths labelled APB and AQB joining A to B (see Figure 8). The value of the line integral along each of these two paths is the same: as we found in Example 4 it is

$$\left(\frac{x_2^3}{3} + \frac{y_2^2}{2}\right) - \left(\frac{x_1^3}{3} + \frac{y_1^2}{2}\right).$$

Hence the value of the line integral of $x^2\mathbf{i} + y\mathbf{j}$ round the closed curve $APBQA$ is zero because

$$\int_{APBQA} \mathbf{F}\cdot d\mathbf{r} = \int_{APB} \mathbf{F}\cdot d\mathbf{r} + \int_{BQA} \mathbf{F}\cdot d\mathbf{r} = \int_{APB} \mathbf{F}\cdot d\mathbf{r} - \int_{AQB} \mathbf{F}\cdot d\mathbf{r} = 0,$$

and this will be true for every closed path. The vector field $\mathbf{F} = x^2\mathbf{i} + y\mathbf{j}$ is an example of a **conservative** vector field.

Exercise 6

(i) Show that the vector field grad ϕ where $\phi = \dfrac{x^3}{3} + \dfrac{y^2}{2}$ is the same as the vector field given in Example 4.

(ii) Show that for the vector field grad ϕ in part (i) and the points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$,

$$\int_{AB} \text{grad}\,\phi\cdot d\mathbf{r} = \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1).$$

[*Solution on p. 46*]

Exercise 7

Use the same method as in Example 4 to show that the value of the line integral of the vector field grad ϕ for any scalar field is given by $\phi(B) - \phi(A)$, where A and B are the two end points of any path in the domain of ϕ.

[*Solution on p. 46*]

The result of Exercise 7 shows that the vector field grad ϕ obtained from a scalar field ϕ is a conservative vector field. We shall continue the discussion of conservative vector fields in Subsection 4.3 of this unit.

Summary of Section 3

1. The **scalar line integral** is of the form $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}$ is a vector field function and C is a curve along which the integration is carried out. If C is a closed curve then we write $\oint_C \mathbf{F}\cdot d\mathbf{r}$.
2. A curve can be expressed in terms of a parameter t by the **parametric equations**
$$x = x(t), \qquad y = y(t), \qquad z = z(t) \qquad (a \leqslant t \leqslant b).$$
3. To evaluate a scalar line integral we express $\mathbf{F}$ and $\mathbf{r}$ in terms of a parameter t and then the line integral $\int \mathbf{F}\cdot d\mathbf{r}$ can be written in terms of an ordinary integral in t, as summarized in the following procedure:

Procedure 3.4: Evaluating line integrals

Step 1. *Parametrize:* find a parametrization of the curve so that every point on the curve is given in terms of a parameter t.

Step 2. *Substitute:* write each variable x, y, z, the vector field $\mathbf{F}$ and $\dfrac{d\mathbf{r}}{dt}$ in terms of t and write the line integral $\int \mathbf{F}\cdot d\mathbf{r}$ as an integral in terms of t.

Step 3. *Evaluate:* evaluate the t integral.

---

4. The line integral of the gradient of a scalar field ϕ is independent of the path. We have $\int_{AB} \nabla\phi\cdot d\mathbf{r} = \phi(B) - \phi(A)$.
5. A vector field $\mathbf{F}$ is **conservative** if for every closed path C drawn in the domain of $\mathbf{F}$ we have
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0.$$
Otherwise $\mathbf{F}$ is a **non-conservative** field.

4 The curl of a vector field

4.1 Line integrals and curl (Television Subsection)

The television programme investigates line integrals of velocity fields representing three fluid flows and uses the line integral of a vector field $\mathbf{u}$ to define a new vector field called curl $\mathbf{u}$.

During the television programme the results of the following exercises are used; so work through the exercises before watching the programme.

Exercise 1

A circular disc rotates about its centre O with constant angular velocity ω; that is to say, the angle turned through in time t is ωt (radians). Any point P on the disc has velocity of magnitude ωa and in a direction perpendicular to the radius vector $\overrightarrow{OP}$ where a is the magnitude of $\overrightarrow{OP}$ (see Figure 1).

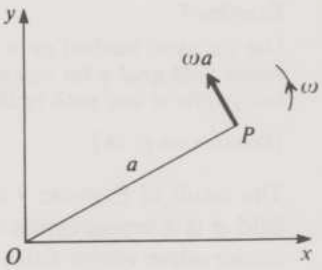


Figure 1

(i) Show that the velocity vector field $\mathbf{u}$ is given by $\mathbf{u} = \omega(-y\mathbf{i} + x\mathbf{j})$ using Cartesian coordinates with origin at O.

(ii) Evaluate the line integral $\oint_C \mathbf{u} \cdot d\mathbf{r}$ where C is a circle whose centre is the origin and whose radius is b.

[*Solution on p. 46*]

Exercise 2

If $f(y)$ is a differentiable function of y, show, by using Taylor series, that

$$\lim_{\delta y \to 0} \frac{f\left(y - \frac{\delta y}{2}\right) - f\left(y + \frac{\delta y}{2}\right)}{\delta y} = -\frac{df}{dy}.$$

If you find this exercise difficult, consult the solution.

[*Solution on p. 46*]

The programme can be summarized as follows.

Part 1: we begin by introducing two different flows of a liquid:

(i) the motion of water in a bowl which is rotating with constant angular velocity, and

(ii) the motion of water in a vortex tank, which is rather like what happens to bath water near to a plughole.

These two flows appear similar because the motion is circular; but small floats placed in the fluid behave differently in the two cases.

Part 2: the rotation of a rigid disc is used as a simple model for the motion in the bowl and we find the velocity field $\mathbf{u}$ representing this flow. We then use a line integral to define the z component of a new vector field called curl $\mathbf{u}$. The expression defining $(\text{curl}\,\mathbf{u})_z$ relates the line integral to the rotational properties of the vector field $\mathbf{u}$. The value of $(\text{curl}\,\mathbf{u})_z$ is used to predict whether small floats placed in the fluid flows will rotate or not.

Part 3: the flow in a long rectangular tank is then investigated. At first sight there appears to be no rotational motion of the fluid in the tank and we would expect a float not to rotate. However the formula for $(\text{curl}\,\mathbf{u})_z$ predicts otherwise. Have we made a mistake?

Now watch the television programme and find out.

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In the television programme we defined the z component of the curl of a two-dimensional vector field in the x,y-plane. We can define the curl for any such vector field in the following way:

Suppose that $\mathbf{F}$ is a two-dimensional vector field defined over a region of the x,y-plane. Then the z component of the vector field curl $\mathbf{F}$ at a point P in the domain of $\mathbf{F}$, is defined by

$$(\text{curl}\,\mathbf{F})_z = \lim_{A \to 0} \left(\frac{1}{A} \oint_C \mathbf{F} \cdot d\mathbf{r} \right)$$

where C is a curve drawn in the domain of $\mathbf{F}$ surrounding the point P and A is the area within C (see Figure 2).

The result is the same regardless of the shape of the curve C.

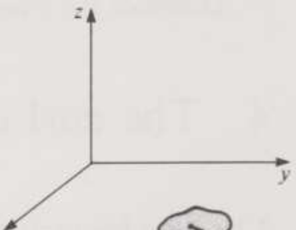


Figure 2

During the programme the results of several calculations involving the limit in this definition above were assumed. In the following exercises you are asked to show that these assumptions are correct.

Exercise 3

The velocity field for a disc rotating about the origin with constant angular velocity ω is given by $\mathbf{u} = -\omega y\mathbf{i} + \omega x\mathbf{j}$. Show that $\lim_{A\to 0}\left(\frac{1}{A}\oint_C \mathbf{u}\cdot d\mathbf{r}\right) = 2\omega$ in each of the following cases:

(i) C is a square of side b, and centred at the origin (see Figure 3);

(ii) C is a square of side b, and centred at a point (x_0, y_0) (see Figure 4).

[*Solution on p. 46*]

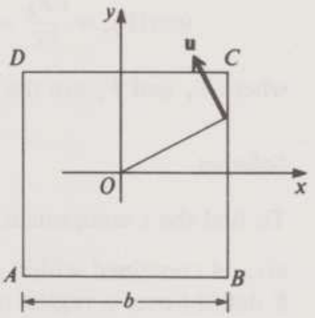


Figure 3

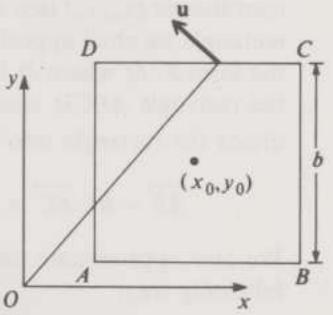


Figure 4

Exercise 4

The velocity vector field for the flow in the vortex tank is given by

$$\mathbf{u} = \frac{c}{d}\mathbf{e}_\theta$$

where $\mathbf{e}_\theta$ is a unit vector perpendicular to the radius vector at a point P (see Figure 5) distance d from the origin.

(i) Show that $\mathbf{e}_\theta = \dfrac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}$.

(ii) Hence show that $\mathbf{u} = \dfrac{c}{(x^2 + y^2)}(-y\mathbf{i} + x\mathbf{j})$.

(iii) Evaluate the line integral of $\mathbf{u}$ round the curve $C = ABCD$ shown in Figure 6, where AB and CD are parts of radii and BC and DA are arcs of circles.

(iv) Hence evaluate $(\operatorname{curl}\mathbf{u})_z$ at the point P.

[*Solution on p. 47*]

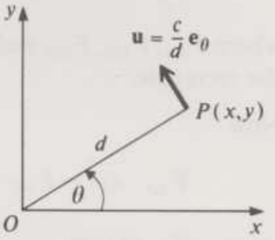


Figure 5

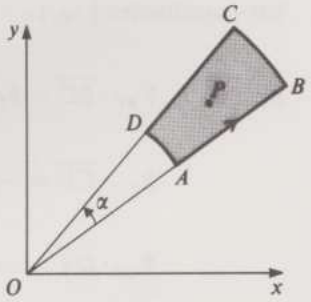
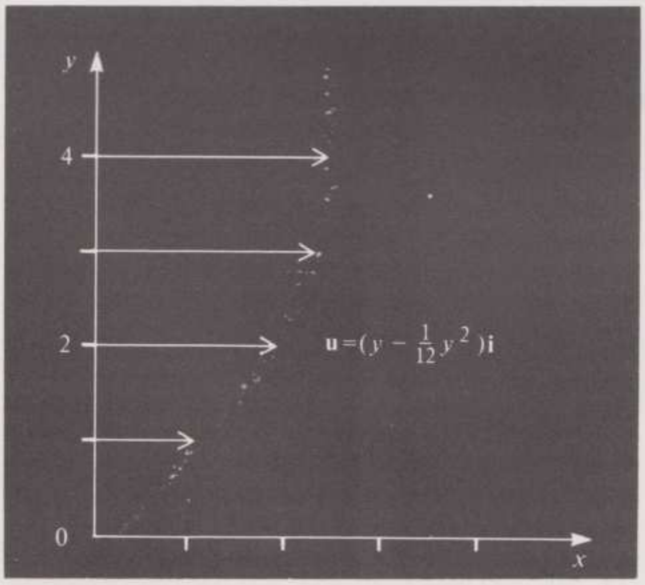


Figure 6

We saw in the television programme that the curl of a velocity field $\mathbf{u}$ describes the rotational properties of the field. We could predict whether a small float placed in a fluid would rotate or not depending on whether the value of $(\operatorname{curl}\mathbf{u})_z$ was non-zero or zero respectively.

The mathematical analysis for the third fluid experiment, involving the linear tank, introduced a different way of calculating curl. We found that for the velocity field $\mathbf{u} = f(y)\mathbf{i}$, the z component of $\operatorname{curl}\mathbf{u}$ could be expressed as $-\dfrac{df}{dy}$, i.e. in terms of the rate of change of the velocity field.



Calculating derivatives is certainly easier than evaluating line integrals and taking limits. In the following example we obtain a formula for $(\operatorname{curl}\mathbf{F})_z$ for the more general two-dimensional vector field $\mathbf{F} = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$, in terms of the spatial rates of change of the components of $\mathbf{F}$.

Example 1

Suppose that $\mathbf{F}$ is a two-dimensional vector field defined over a region of the x, y-plane. By integrating round a small rectangle show that

$$(\text{curl}\,\mathbf{F})_z = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

where F_1 and F_2 are the x and y components of the vector field.

Solution

To find the z component of curl**F** we need to evaluate $\lim_{A \to 0} \frac{1}{A} \oint_C \mathbf{F} \cdot d\mathbf{r}$ for some area A contained within a closed curve C. Consider a two-dimensional vector field **F** defined over a region of the x,y-plane and let $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j}$. For the curve C, consider a small rectangle $ABCD$ with sides h and l and centre at the point P with coordinates (x_0, y_0) (see Figure 7). To evaluate the line integral of **F** round the rectangle we shall appeal to the definition of a line integral as the sum of terms of the form $\mathbf{F} \cdot \delta\mathbf{r}$ where $\delta\mathbf{r}$ is a small displacement along the curve of integration. For the rectangle $ABCD$, since we are assuming that h and l are small, suppose that we divide the rectangle into four small displacements $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$ where

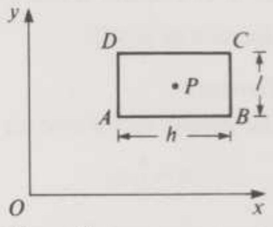


Figure 7

$$\overrightarrow{AB} = h\,\mathbf{i},\ \overrightarrow{BC} = l\,\mathbf{j},\ \overrightarrow{CD} = -h\,\mathbf{i} \text{ and } \overrightarrow{DA} = -l\,\mathbf{j}.$$

We now approximate the line integral as the sum of four scalar products in the following way:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \simeq \mathbf{F}_{AB} \cdot \overrightarrow{AB} + \mathbf{F}_{BC} \cdot \overrightarrow{BC} + \mathbf{F}_{CD} \cdot \overrightarrow{CD} + \mathbf{F}_{DA} \cdot \overrightarrow{DA}$$

where $\mathbf{F}_{AB}$, $\mathbf{F}_{BC}$, $\mathbf{F}_{CD}$ and $\mathbf{F}_{DA}$ are the values of **F** at the mid-points of the sides of the rectangle.

Now

$$\mathbf{F}_{AB} \cdot \overrightarrow{AB} = \mathbf{F}_{AB} \cdot h\,\mathbf{i} = F_1\left(x_0, y - \frac{l}{2}\right)h$$

because $\mathbf{F}_{AB} \cdot \mathbf{i}$ is the x-component of **F** evaluated at the mid-point of AB, which has coordinates $\left(x_0, y_0 - \frac{l}{2}\right)$.

Similarly $\mathbf{F}_{BC} \cdot \overrightarrow{BC} = F_2\left(x_0 + \frac{h}{2}, y_0\right)l$

$$\mathbf{F}_{CD} \cdot \overrightarrow{CD} = -F_1\left(x_0, y_0 + \frac{l}{2}\right)h$$

$$\mathbf{F}_{DA} \cdot \overrightarrow{DA} = -F_3\left(x_0 - \frac{h}{2}, y_0\right)l$$

and so we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \simeq \left(F_1\left(x_0, y_0 - \frac{l}{2}\right) - F_1\left(x_0, y_0 + \frac{l}{2}\right)\right)h + \left(F_2\left(x_0 + \frac{h}{2}, y_0\right) - F_2\left(x_0 - \frac{h}{2}, y_0\right)\right)l.$$

The area contained within C is the area of the rectangle $ABCD$ and this is hl. Thus

$$\frac{1}{A}\oint_C \mathbf{F} \cdot d\mathbf{r} \simeq \frac{F_1\left(x_0, y_0 - \frac{l}{2}\right) - F_1\left(x_0, y_0 + \frac{l}{2}\right)}{l} + \frac{F_2\left(x_0 + \frac{h}{2}, y_0\right) - F_2\left(x_0 - \frac{h}{2}, y_0\right)}{h}$$

Now in the limit as $A \longrightarrow 0$, both $h \longrightarrow 0$ and $l \longrightarrow 0$; and we have

$$\lim_{h,l\to 0}\frac{F_1\left(x_0,y_0-\dfrac{l}{2}\right)-F_1\left(x_0,y_0+\dfrac{l}{2}\right)}{l}=-\frac{\partial F_1}{\partial y}$$

and

$$\lim_{h,l\to 0}\frac{F_2\left(x_0+\dfrac{h}{2},y_0\right)-F_2\left(x_0-\dfrac{h}{2},y_0\right)}{h}=\frac{\partial F_2}{\partial x}.$$

These results follow either by expanding the numerator as a Taylor series about (x_0, y_0) and then taking the limit as $h, l \longrightarrow 0$, or from the definition of partial differentiation in *Unit 25*.

Finally we can write $\lim\limits_{A\to 0}\dfrac{1}{A}\oint \mathbf{F}\cdot d\mathbf{r} = -\dfrac{\partial F_1}{\partial y}+\dfrac{\partial F_2}{\partial x}$ and so

$$\boxed{(\operatorname{curl}\mathbf{F})_z=\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)}$$

for a two-dimensional vector field.

Note that if $F_2 = 0$ then $(\operatorname{curl}\mathbf{F})_z = -\dfrac{\partial F_1}{\partial y}$, which is in agreement with the result obtained in the television programme for the velocity field of the flow in the linear tank.

We now have a formula for the z component of $\operatorname{curl}\mathbf{F}$ in terms of the derivatives of the components of $\mathbf{F}$. In the next exercise you can check the results of Exercises 3 and 4 by using this derivative formula.

Exercise 5

Use the derivative formula to find the value of $(\operatorname{curl}\mathbf{u})_z$ for the velocity vector fields

(i) $\mathbf{u} = -\omega y\mathbf{i} + \omega x\mathbf{j}$;

(ii) $\mathbf{u} = \dfrac{c}{(x^2+y^2)}(-y\mathbf{i} + x\mathbf{j})$.

In each case compare your answer with the relevant result obtained in Exercise 3 or 4.

[*Solution on p. 47*]

The velocity field given in the first part of this exercise represents the motion of the rotating disc. We find that $(\operatorname{curl}\mathbf{u})_z = 2\omega$ at every point on the disc, and that is the result we obtained in the television programme by using line integrals. The velocity field given in the second part of this exercise represents the flow in the vortex tank. We find that $(\operatorname{curl}\mathbf{u})_z = 0$ in this case, which again agrees with the line integral calculation.

So the curl of a vector field describes the rotational properties of the vector field. Its effect can be illustrated by placing small floats into a moving liquid. Then the curl of the velocity field at a particular point is twice the angular velocity of a float placed at that point.

4.2 Curl of a three-dimensional vector field

The television programme and the definition of curl in the last subsection considered only two-dimensional vector fields defined over regions of the x,y-plane. We defined the z component of curl for a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ by the line integral

$$\lim_{A\to 0}\frac{1}{A}\oint_C \mathbf{F}\cdot d\mathbf{r}.$$

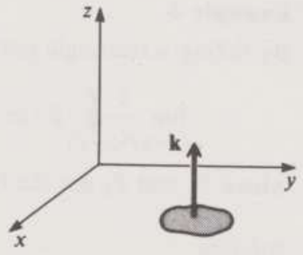


Figure 8

Now the z direction is normal to the area A (see Figure 8); and the unit vector $\mathbf{k}$ points in the z direction so we could have written $(\operatorname{curl}\mathbf{F})_z$ as $(\operatorname{curl}\mathbf{F})\cdot\mathbf{k}$.

We define curl for a three-dimensional vector field in an analogous way.

Suppose that **F** is a three-dimensional vector field defined over a region of space. Then there exists a vector field curl**F** with the following property: for any point P and any unit vector **n**, the component of curl**F** in the direction of **n** is given by

$$(\text{curl}\mathbf{F})_{\mathbf{n}} = (\text{curl}\mathbf{F})\cdot\mathbf{n} = \lim_{A\to 0}\oint_C \mathbf{F}\cdot d\mathbf{r}$$

where C is a small curve lying in a plane and enclosing the point P such that **n** is normal to this plane containing C and A is the area of the plane enclosed within C (see Figure 9).

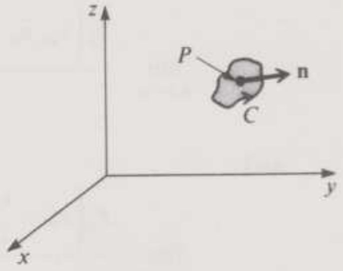


Figure 9

When **n** is the unit vector **k**, then $(\text{curl}\mathbf{F})_{\mathbf{k}}$ is just what we denoted previously by $(\text{curl}\mathbf{F})_z$.

This is rather a formal definition which is a simple extension of the definition of curl for a two-dimensional vector field. You are more familiar with vector fields being given in terms of a vector function and its domain. The domain of curl**F** is the same as the domain of **F** because curl is defined for the same set of points as is **F**. To write curl**F** as a vector function of the form $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ we need the components of curl in the directions of **i**, **j** and **k**. Now we can find these components by using the definition above:

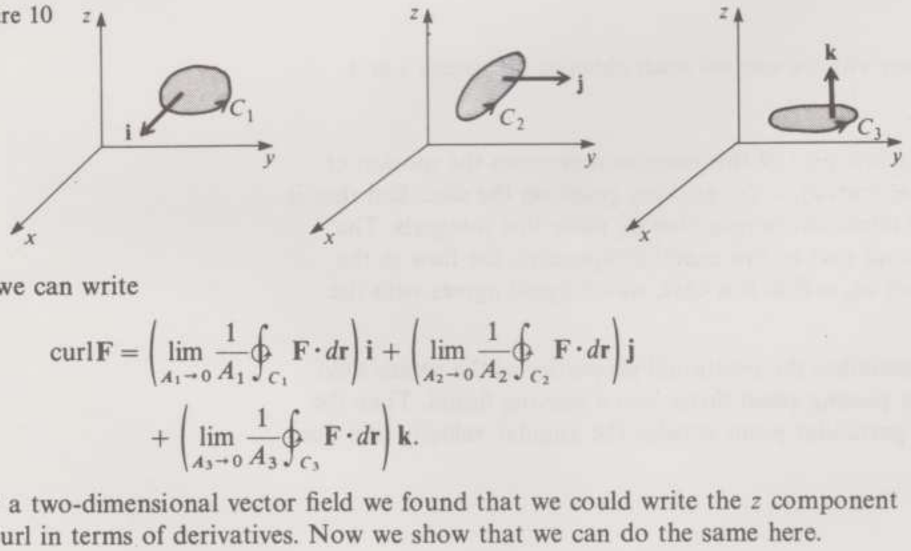
$$a_1 = (\text{curl}\mathbf{F})\cdot\mathbf{i} = \lim_{A_1\to 0}\frac{1}{A_1}\oint_{C_1}\mathbf{F}\cdot d\mathbf{r},$$

$$a_2 = (\text{curl}\mathbf{F})\cdot\mathbf{j} = \lim_{A_2\to 0}\frac{1}{A_2}\oint_{C_2}\mathbf{F}\cdot d\mathbf{r}$$

and $$a_3 = (\text{curl}\mathbf{F})\cdot\mathbf{k} = \lim_{A_3\to 0}\frac{1}{A_3}\oint_{C_3}\mathbf{F}\cdot d\mathbf{r}$$

where C_1, C_2 and C_3 are curves drawn parallel to the y,z-plane, x,z-plane and x,y-plane respectively, and A_1, A_2 and A_3 are the areas contained within C_1, C_2 and C_3 respectively (see Figure 10).

Figure 10



So we can write

$$\text{curl}\mathbf{F} = \left(\lim_{A_1\to 0}\frac{1}{A_1}\oint_{C_1}\mathbf{F}\cdot d\mathbf{r}\right)\mathbf{i} + \left(\lim_{A_2\to 0}\frac{1}{A_2}\oint_{C_2}\mathbf{F}\cdot d\mathbf{r}\right)\mathbf{j}$$
$$+\left(\lim_{A_3\to 0}\frac{1}{A_3}\oint_{C_3}\mathbf{F}\cdot d\mathbf{r}\right)\mathbf{k}.$$

For a two-dimensional vector field we found that we could write the z component of curl in terms of derivatives. Now we show that we can do the same here.

Example 2

By taking a rectangle parallel to the x,y-plane show that

$$\lim_{A_3\to 0}\frac{1}{A_3}\oint_{C_3}\mathbf{F}\cdot d\mathbf{r} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

where F_1 and F_2 are the **i** and **j** components of **F** respectively.

Solution

Consider a vector field **F** and let

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}.$$

For the curve C_3 take a small rectangle $ABCD$ with sides h and l and centre at the point P with coordinates (x_0, y_0, z_0), the rectangle being drawn in the plane $z = z_0 =$ constant (see Figure 11).

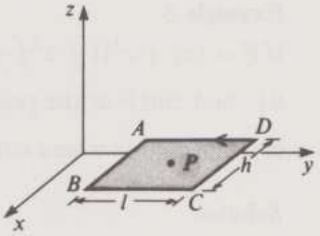


Figure 11

The analysis is similar to that used in the solution to Example 1. We write the line integral $\oint_{C_3} \mathbf{F}\cdot d\mathbf{r}$ as the sum of four scalar products as follows:

$$\oint_{C_3} \mathbf{F}\cdot d\mathbf{r} \simeq \mathbf{F}_{AB}\cdot\overrightarrow{AB} + \mathbf{F}_{BC}\cdot\overrightarrow{BC} + \mathbf{F}_{CD}\cdot\overrightarrow{CD} + \mathbf{F}_{DA}\cdot\overrightarrow{DA}.$$

For this vector field $\mathbf{F}$ we have three components; however the four scalar products involve multiplying by either $\mathbf{i}$ or $\mathbf{j}$ so that the $\mathbf{k}$ component of $\mathbf{F}$, i.e. F_3, does not appear in the value of the line integral. For example,

$$\mathbf{F}_{AB}\cdot\overrightarrow{AB} = \mathbf{F}_{AB}\cdot(h\mathbf{i}) = F_1\left(x_0, y_0 - \frac{l}{2}, z_0\right)h,$$

and $$\mathbf{F}_{BC}\cdot\overrightarrow{BC} = \mathbf{F}_{BC}\cdot(l\mathbf{j}) = F_2\left(x_0 + \frac{h}{2}, y_0, z_0\right)l.$$

$$\text{Thus } \frac{1}{A_3}\oint_{C_3} \mathbf{F}\cdot d\mathbf{r} \simeq \frac{F_1\left(x_0, y_0 - \frac{l}{2}, z_0\right) - F_1\left(x_0, y_0 + \frac{l}{2}, z_0\right)}{l}$$

$$+ \frac{F_2\left(x_0 + \frac{h}{2}, y_0, z_0\right) - F_2\left(x_0 - \frac{h}{2}, y_0, z_0\right)}{h}$$

and in the limit as $A_3 \longrightarrow 0$ (i.e. $h \longrightarrow 0$ and $l \longrightarrow 0$) we have

$$\lim_{A_3 \to 0} \frac{1}{A_3}\oint_{C_3} \mathbf{F}\cdot d\mathbf{r} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

and this is the $\mathbf{k}$ component of curl$\mathbf{F}$. In a similar way we can obtain derivative forms for the $\mathbf{i}$ and $\mathbf{j}$ components of curl$\mathbf{F}$. For the $\mathbf{i}$ component we choose for C_1 a small rectangle in the $x = x_0$ plane (see Figure 12). The $\mathbf{i}$ component becomes $\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}$. For the $\mathbf{j}$ component we choose for C_2 a small rectangle in the $y = y_0$ plane (see Figure 13). The $\mathbf{j}$ component becomes $\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}$.

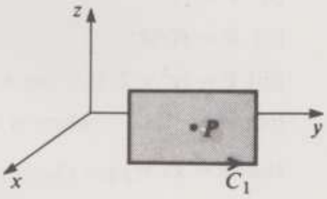


Figure 12

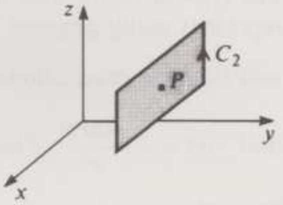


Figure 13

Thus we can write the vector field curl$\mathbf{F}$ in a form that is easy to evaluate, namely:

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

Applying this formula to the two-dimensional vector field $\mathbf{F} = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$ we obtain the same result as in the last subsection; that is $(\text{curl}\,\mathbf{F})_z = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$, but now we can go one stage further and write curl$\mathbf{F}$ as the vector

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

We were not able to do this before because we knew nothing about the x and y components of curl$\mathbf{F}$. However we could have made an educated guess at their values. We saw in the television programme that for the rotating disc, $(\text{curl}\,\mathbf{u})_z = 2\omega$. Now angular velocity is a vector quantity; for the rotation of the x, y-plane the angular velocity vector is $\omega\mathbf{k}$, so we might have suspected that curl$\,\mathbf{u}$ is also in the $\mathbf{k}$ direction.

Example 3

If $\mathbf{F} = (xy + z^2)\mathbf{i} + x^2\mathbf{j} + (xz - 2)\mathbf{k}$

(i) find $\operatorname{curl}\mathbf{F}$ at the point $(1, -1, 3)$;

(ii) find points where $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

Solution

We use the differential formula to calculate $\operatorname{curl}\mathbf{F}$.

$$F_1 = xy + z^2, \qquad F_2 = x^2, \qquad F_3 = xz - 2,$$

so that $\dfrac{\partial F_1}{\partial y} = x$ and $\dfrac{\partial F_1}{\partial z} = 2z$,

$$\frac{\partial F_2}{\partial x} = 2x \text{ and } \frac{\partial F_2}{\partial z} = 0,$$

$$\frac{\partial F_3}{\partial x} = z \text{ and } \frac{\partial F_3}{\partial y} = 0,$$

giving $\operatorname{curl}\mathbf{F} = (0)\mathbf{i} + (2z - z)\mathbf{j} + (2x - x)\mathbf{k}$.

(i) At the point $(1, -1, 3)$ we have $\operatorname{curl}\mathbf{F} = 3\mathbf{j} + \mathbf{k}$.

(ii) $\operatorname{curl}\mathbf{F}$ is zero when $x = 0$ and $z = 0$ for any value of y.

Exercise 6

Find $\operatorname{curl}\mathbf{F}$ if

(i) $\mathbf{F} = \mathbf{r}$;

(ii) $\mathbf{F} = f(r)\hat{\mathbf{r}}$;

(iii) $\mathbf{F} = (y^2 + 2z)\mathbf{i} + (xy + 6z)\mathbf{j} + (z^2 + 2xz + y)\mathbf{k}$;

(iv) $\mathbf{F} = \operatorname{grad}\phi$ where ϕ is a scalar field.

Here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ and $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$.

[*Solution on p. 48*]

The type of vector field function in part (ii) of this exercise can be used to represent many physical quantities in nature. For example, you will see in a later unit that the gravitational force field can be written in the form $\mathbf{F}_g = -\dfrac{GMm}{|\mathbf{r}|^2}\hat{\mathbf{r}}$ so that $f(r) = -\dfrac{GMm}{r^2}$. The result of this exercise shows that $\operatorname{curl}\mathbf{F}_g = 0$.

We will see in the next subsection that if the curl of a vector field is zero for all points in its domain, then the vector field is conservative, so we will use curl mainly as a test for conservative fields.

The derivative formula for $\operatorname{curl}\mathbf{F}$ contains terms involving partial derivatives. The expression is not particularly easy to remember; however the following exercise shows an alternative notation in terms of the differential operator del.

Exercise 7

By expanding the cross product of ∇ and $\mathbf{F}$ show that $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$.

[*Solution on p. 48*]

This is an alternative definition for the curl of a vector field which you may come across in other texts. The way we have defined the curl in this section more readily shows its important physical significance.

4.3 Conservative fields

The vector field $\mathbf{F} = \dfrac{C(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2 + y^2 + z^2)^{3/2}}$ is one that we have considered at various times in this unit. In Example 2 of Section 1 we saw that this vector field function represents the gravitational force on a body a distance r from the Earth's centre. In Example 3 of Section 2 we showed that it could be derived from the scalar field $\phi = \dfrac{C}{\sqrt{x^2 + y^2 + z^2}}$. In Exercise 6 of Section 3 we saw that the line integral of $\mathbf{F}$ between two points A and B is independent of the path taken. And Exercise 5(ii) of this section shows that $\operatorname{curl}\mathbf{F} = \mathbf{0}$. Now we shall put these ideas together by talking about a special class of vector fields called *conservative fields* for which these last three properties are true.

On the audio-tape we defined a conservative vector field $\mathbf{F}$ as one for which the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed curve C drawn in the domain of $\mathbf{F}$. (The reason for the name has to do with the conservation of energy for bodies acted on by forces which are modelled by conservative vector fields.) If a vector field does not have this property then it is *non-conservative*. The velocity field $\mathbf{u}$ representing the rotating disc is an example of a non-conservative vector field, because when C is a circle of radius b and centre the origin,

$$\oint_C \mathbf{u}\cdot d\mathbf{r} = 2\omega\pi b^2 \qquad \text{(see Exercise 1).}$$

This definition of a conservative vector field has an important consequence for the curl of $\mathbf{F}$. We have seen that

$$\operatorname{curl}\mathbf{F} = \lim_{A_1\to 0}\left(\frac{1}{A_1}\oint_{C_1}\mathbf{F}\cdot d\mathbf{r}\right)\mathbf{i} + \lim_{A_2\to 0}\left(\frac{1}{A_2}\oint_{C_2}\mathbf{F}\cdot d\mathbf{r}\right)\mathbf{j} + \lim_{A_3\to 0}\left(\frac{1}{A_3}\oint_{C_3}\mathbf{F}\cdot d\mathbf{r}\right)\mathbf{k}.$$

Now for a conservative vector field, the line integral round every closed curve is zero. This means that each of the components of $\operatorname{curl}\mathbf{F}$ is zero, and we have the important result:

If $\mathbf{F}$ is a conservative vector field then $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

The converse of this is also true: if $\operatorname{curl}\mathbf{F} = \mathbf{0}$ and the domain of $\mathbf{F}$ is sufficiently simple then $\mathbf{F}$ is conservative; however we will not be able to prove this statement in this course. This gives an alternative definition of a conservative vector field which you may come across in other texts.

We shall not say precisely what we mean by 'sufficiently simple'. Certainly every domain we consider in this unit is sufficiently simple in this context.

If $\operatorname{curl}\mathbf{F} = \mathbf{0}$ at every point of the domain of $\mathbf{F}$, then $\mathbf{F}$ is said to be a conservative vector field.

Thus if we want to decide whether a vector field $\mathbf{F}$ is conservative or not, instead of evaluating line integrals round every possible closed curve that could be drawn in the domain of the vector field, we evaluate $\operatorname{curl}\mathbf{F}$ in its derivative form and see if its value is zero or not.

Example 4

Which of the following vector fields is conservative?

(i) $\mathbf{F} = zx^2\,\mathbf{i} + zy^2\,\mathbf{j} + \frac{1}{3}(x^3 + y^3)\,\mathbf{k}$;

(ii) $\mathbf{G} = x\,\mathbf{i} + x^2y\,\mathbf{j}$.

Solution

We have to evaluate $\operatorname{curl}\mathbf{F}$ and $\operatorname{curl}\mathbf{G}$. Using the derivative formula for curl gives:

(i) $\operatorname{curl}\mathbf{F} = (y^2 - y^2)\mathbf{i} + (x^2 - x^2)\mathbf{j} + (0 - 0)\mathbf{k}$

$\qquad = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$

Thus $\operatorname{curl}\mathbf{F} = \mathbf{0}$ so that $\mathbf{F}$ is a conservative vector field;

(ii) $$\begin{aligned}\operatorname{curl}\mathbf{G} &= (0-0)\mathbf{i} + (0-0)\mathbf{j} + (2xy-1)\mathbf{k}\\ &= (2xy-1)\mathbf{k}.\end{aligned}$$

Thus $\operatorname{curl}\mathbf{G} \neq \mathbf{0}$ at every point, so $\mathbf{G}$ is a non-conservative vector field.

Exercise 8

(i) The vector field $\mathbf{F} = 2x\mathbf{i} + (xz-2)\mathbf{j} + xy\mathbf{k}$ was shown to be non-conservative in Section 3 (Exercise 3) by evaluating the line integral along a closed path. Check this result by finding the value of $\operatorname{curl}\mathbf{F}$.

(ii) Is the vector field $\mathbf{G} = r^n\hat{\mathbf{r}}$ conservative, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \mathbf{r}/r$?

[*Solution on p. 48*]

Exercise 9

If $\mathbf{F}$ is a conservative vector field show that the line integral of $\mathbf{F}$ between two points A and B, i.e. $\int_{AB} \mathbf{F}\cdot d\mathbf{r}$, is independent of the path taken.

[*Solution on p. 48*]

Exercise 9 shows that for a conservative vector field the value of the line integral $\int_{AB} \mathbf{F}\cdot d\mathbf{r}$ depends only on the end points A and B of the path and not on the path itself. In particular we can define a scalar field ϕ at every point $P = (x, y, z)$ in the domain of the conservative field $\mathbf{F}$ by letting

$$\left.\begin{aligned}&\phi(x,y,z) = \int_{OP} \mathbf{F}\cdot d\mathbf{r} \qquad \text{(see Figure 14).}\\ &\text{Then } \int_{AB} \mathbf{F}\cdot d\mathbf{r} = \phi(B) - \phi(A).\end{aligned}\right\} \quad (1)$$

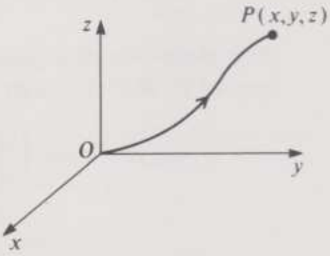


Figure 14

We recall from the result of Exercise 7 in Section 3 that the value of the line integral of $\operatorname{grad}\phi$ along any curve between two points A and B is $\phi(B) - \phi(A)$. That is,

$$\int_{AB} \operatorname{grad}\phi\cdot d\mathbf{r} = \phi(B) - \phi(A).$$

In general it can be shown that when Equations (1) hold then $\mathbf{F} = \operatorname{grad}\phi$ for the scalar field ϕ.

One application of this idea is to the motion of a particle acted upon by a position-dependent force which can be modelled by a conservative vector field. Such a vector field is called a conservative force field. For instance, consider a body moving in a one-dimensional force field $\mathbf{F} = F(x)\mathbf{i}$. Any one-dimensional force field is conservative and the scalar field which we define from this force field is just $\phi(x) = \int_0^x F(s)ds$. In *Unit 4* we defined the potential energy of a body to be $U(x) = -\int_0^x F(s)ds$, so the one-dimensional scalar field that we have introduced here is just minus the potential energy of mechanics.

We can generalize the ideas of one-dimensional mechanics and we use the line integral $-\int \mathbf{F}\cdot d\mathbf{r}$ to be the **potential energy** U of a body moving in a *conservative* force field $\mathbf{F}$. Conversely, given a potential energy function $U(x, y, z)$, the force acting on a particle is

$$\mathbf{F} = -\nabla U.$$

In a later unit we will follow this idea up. You will see that for a *conservative* force field we have a *conservation* of energy equation,

kinetic energy + potential energy = constant,

and thus the name 'conservative'.

Summary of Section 4

1. For a two-dimensional vector field **F** defined over a region of the x,y-plane, the **z component of curl F** at a point P is defined by

$$(\operatorname{curl}\mathbf{F})_z = \lim_{A\to 0}\left(\frac{1}{A}\oint_C \mathbf{F}\cdot d\mathbf{r}\right)$$

where C is a small curve drawn in the domain of **F** surrounding the point P and A is the area within C. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ then an alternative way of writing $(\operatorname{curl}\mathbf{F})_z$ is

$$(\operatorname{curl}\mathbf{F})_z = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

2. For a three-dimensional vector field **F** we define the component of curl**F** at a point P in the direction of a unit vector **n** to be

$$(\operatorname{curl}\mathbf{F})_{\mathbf{n}} = (\operatorname{curl}\mathbf{F})\cdot\mathbf{n} = \lim_{A\to 0}\left(\frac{1}{A}\oint_C \mathbf{F}\cdot d\mathbf{r}\right)$$

where C is a small planar curve enclosing the point P, A is the area enclosed within C and **n** is normal to the plane of the curve C.

3. We evaluate curl of a vector field by using the derivative formula: if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$

then $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$

4. The curl of a vector field **F** is another vector field defined over the same domain as **F**; curl is a detector for rotational motion. For instance, the curl of a fluid velocity field describes the spin of a float placed in the fluid.

5. For a *conservative* vector field:

   (i) $\operatorname{curl}\mathbf{F} = \mathbf{0}$ (and conversely, if the domain of **F** is sufficiently simple);

   (ii) $\int_{AB} \mathbf{F}\cdot d\mathbf{r} = \phi(B) - \phi(A)$ where $\mathbf{F} = \operatorname{grad}\phi$.

   The curl of a vector field **F** can be used as a test to see if **F** is conservative.

Appendix: Solutions to the exercises

Solutions to the exercises in Section 1

1. The temperature T at a point $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ on the metal sheet is defined by $T \propto \log_e r$. But $T = 0$ when $r = 1$, and $T = 100$ when $r = 4$. Putting $T = a(\log_e r) - b$ we get $b = 0$ and $a = 100/\log_e 4$. Thus the scalar field function is

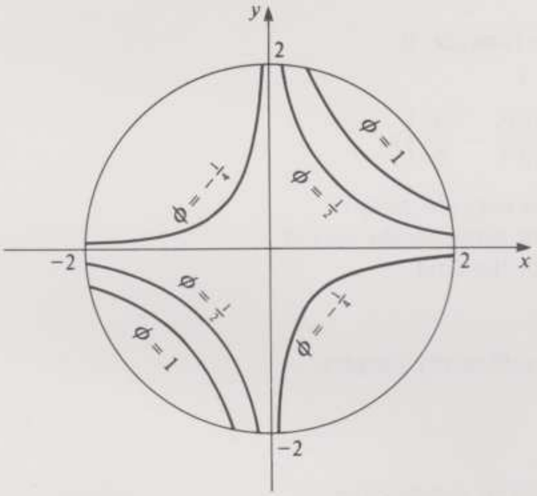
$$T = \frac{100 \log_e r}{\log_e 4}.$$

Since $r = \sqrt{x^2 + y^2}$, the scalar field is defined over the region of the plane given by the set of points (x, y) such that $1 \le \sqrt{x^2 + y^2} \le 4$.

2. The contour curves are given by

$$xy = \text{constant} = c$$

$$\text{i.e. } y = \frac{c}{x}.$$



The figure shows the contour curves for $c = \frac{1}{2}$, $c = -\frac{1}{4}$ and $c = 1$.

3. **(i)(a)** There is a hill on the left, near 'contour 100'.

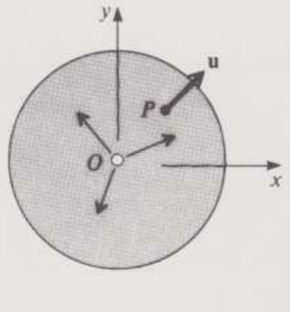
(b) There may be a lake where the contours are marked with negative numbers indicating land levels beneath sea level.

(ii) The land should be fairly level on the right; the contour curves join places at levels 0, 1 and 2 over quite a large area.

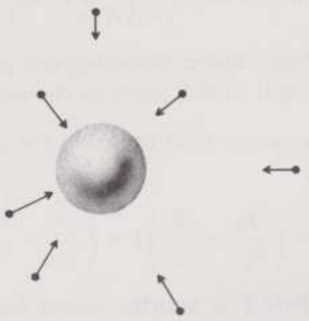
(iii) If we start off at point A and walk in the direction of the arrow the land begins to climb very quickly; the contours are close together so the heights change very quickly indicating a hard climb!

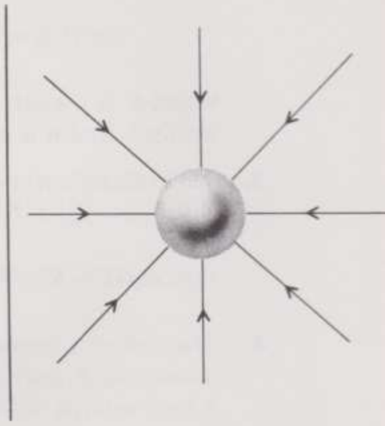
4. The vector quantity is the velocity of the water at points on the plate. The vector field function $\mathbf{u}$ (i.e. the velocity field) is defined for points P on the plate. The region over which the vector field is defined is the set of points (x, y) inside the circle $x^2 + y^2 = 100$ and excluding the origin. The velocity vector field $\mathbf{u}$ is directed along $\overrightarrow{OP}$ and has magnitude $C/|\overrightarrow{OP}|$ for some constant C. So

$$\mathbf{u}(x, y) = \frac{C\overrightarrow{OP}}{|\overrightarrow{OP}||\overrightarrow{OP}|} = \frac{C}{(x^2 + y^2)}(x\mathbf{i} + y\mathbf{j}).$$



5. The Earth's gravitational force field is directed towards the centre of the earth. The left-hand figure shows arrows representing the gravitational force at various points. The field lines are shown in the right-hand figure. The arrowheads indicate the direction of the force field.





6. If you look back at Exercise 4 the field lines in this exercise would describe the velocity vector field of that exercise. So the direction of the vector at each point is in the direction of $x\mathbf{i} + y\mathbf{j}$. The vector field function has the form $\mathbf{F} = f(x, y)(x\mathbf{i} + y\mathbf{j})$ where $f(x, y)$ is some function of x and y taking positive values only.

Solutions to the exercises in Section 2

1. $\phi = x^2 y$;

$$\frac{\partial\phi}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial\phi}{\partial x} = -4 \quad \text{at } (-1, 2);$$

$$\frac{\partial\phi}{\partial y} = x^2 \quad \text{and} \quad \frac{\partial\phi}{\partial y} = 1 \quad \text{at } (-1, 2).$$

So

$$\operatorname{grad}\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} = -4\mathbf{i} + \mathbf{j} \text{ at } (-1, 2).$$

2. The curve $x^2 - 2xy + y^2 = 9$ is a contour of the scalar field $\phi = x^2 - 2xy + y^2$. The vector $\operatorname{grad}\phi$ evaluated at $(0, 3)$ is normal to this scalar field;

$$\operatorname{grad}\phi = (2x - 2y)\mathbf{i} + (-2x + 2y)\mathbf{j}$$

and at $(0, 3)$, $\operatorname{grad}\phi = -6\mathbf{i} + 6\mathbf{j}$, and this is the vector required.

3. $\phi = x^2 y + z^2 x$;

$$\frac{\partial\phi}{\partial x} = 2xy + z^2 \quad \text{and} \quad \frac{\partial\phi}{\partial x} = -2 \quad \text{at } (-1, 1, 0);$$

$$\frac{\partial\phi}{\partial y} = x^2 \quad \text{and} \quad \frac{\partial\phi}{\partial y} = 1 \quad \text{at } (-1, 1, 0);$$

$$\frac{\partial\phi}{\partial z} = 2xz \quad \text{and} \quad \frac{\partial\phi}{\partial z} = 0 \quad \text{at } (-1, 1, 0).$$

So $\text{grad}\,\phi = -2\mathbf{i} + \mathbf{j} + 0\mathbf{k}$ at $(-1, 1, 0)$.

4. (i) $\text{grad}\,\phi = 2xy^2z^2\,\mathbf{i} + 2x^2yz^2\,\mathbf{j} + 2x^2y^2z\,\mathbf{k}$.

The maximum rate of change is $|\text{grad}\,\phi|$ in the direction of $\dfrac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$. At the point $(-1, 1, 1)$ we have

$$|\text{grad}\,\phi| = |-2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}| = 2\sqrt{3}$$

and

$$\frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = -\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}.$$

(ii) The rate of change of ϕ in the direction of $\mathbf{e}$ is $(\text{grad}\,\phi)\cdot\mathbf{e}$. For $\mathbf{e} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$ we have

$$(\text{grad}\,\phi)\cdot\mathbf{e} = \frac{6}{5}xy^2z^2 + \frac{8}{5}x^2y^2z.$$

At the point $(2, 1, -1)$ we have $(\text{grad}\,\phi)\cdot\mathbf{e} = -4$.

5. For two scalar fields ϕ and ψ we would expect that

$$\boldsymbol{\nabla}(\phi\psi) = (\boldsymbol{\nabla}\phi)\psi + \phi(\boldsymbol{\nabla}\psi).$$

To check this result we consider $\boldsymbol{\nabla}(\phi\psi)$ in terms of the partial derivatives.

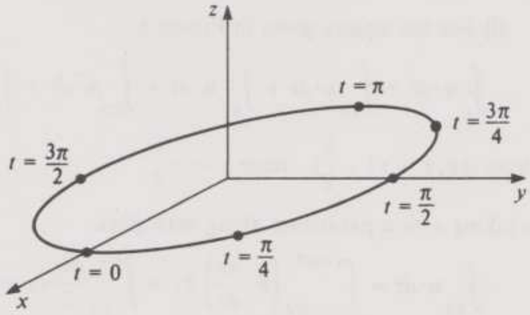
$$\begin{aligned}\boldsymbol{\nabla}(\phi\psi) &= \frac{\partial}{\partial x}(\phi\psi)\mathbf{i} + \frac{\partial}{\partial y}(\phi\psi)\mathbf{j} + \frac{\partial}{\partial z}(\phi\psi)\mathbf{k}\\ &= \left(\frac{\partial\phi}{\partial x}\psi + \phi\frac{\partial\psi}{\partial x}\right)\mathbf{i} + \left(\frac{\partial\phi}{\partial y}\psi + \phi\frac{\partial\psi}{\partial y}\right)\mathbf{j}\\ &\quad + \left(\frac{\partial\phi}{\partial z}\psi + \phi\frac{\partial\psi}{\partial z}\right)\mathbf{k}\\ &= \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\psi\\ &\quad + \phi\left(\frac{\partial\psi}{\partial x}\mathbf{i} + \frac{\partial\psi}{\partial y}\mathbf{j} + \frac{\partial\psi}{\partial z}\mathbf{k}\right)\\ &= (\boldsymbol{\nabla}\phi)\psi + \phi(\boldsymbol{\nabla}\psi).\end{aligned}$$

Solutions to the exercises in Section 3

1.

$t = 0,$	$(a, 0, 0);$
$t = \pi/4,$	$(a/\sqrt{2}, a/\sqrt{2}, 0);$
$t = \pi/2$	$(0, a, 0);$
$t = 3\pi/4,$	$(-a/\sqrt{2}, a/\sqrt{2}, 0);$
$t = \pi,$	$(-a, 0, 0);$
$t = 3\pi/2,$	$(0, -a, 0).$

The curve lies in the plane $z = 0$, and is a circle of radius a with centre at the origin.



2. (i) $x = a\cos t, \quad y = a\sin t;$

$t_1 = 0, \quad t_2 = 2\pi;$

$$\mathbf{u} = \frac{1}{a^2}(a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j});$$

$$\frac{d\mathbf{r}}{dt} = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j};$$

$$\oint_C \mathbf{u}\cdot d\mathbf{r} = \int_0^{2\pi} 0\,dt = 0.$$

(ii) The parametrization is the same, so now

$$\mathbf{u} = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j};$$

$$\frac{d\mathbf{r}}{dt} = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j};$$

$$\oint_C \mathbf{u}\cdot d\mathbf{r} = \int_0^{2\pi} a^2 dt = 2\pi a^2.$$

3. $x = t, \quad y = t^2, \quad z = t^3;$

$t_1 = 0, \quad t_2 = 1;$

$\mathbf{F} = 2t\,\mathbf{i} + (t^4 - 2)\mathbf{j} + t^3\,\mathbf{k};$

$$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k};$$

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (5t^5 - 2t)dt = -\tfrac{1}{6}.$$

4. $x = t, \quad y = t^2, \quad z = t^3;$

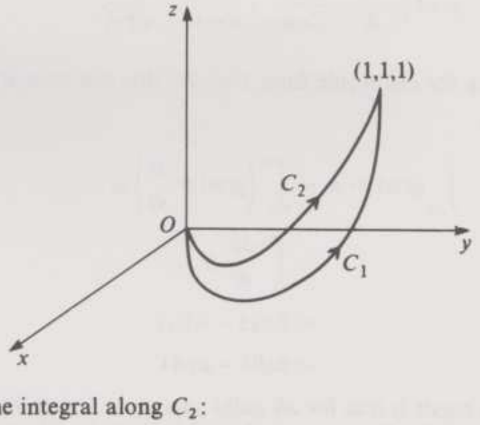
$t_1 = 1, \quad t_2 = 0;$

$\mathbf{F} = 2t\,\mathbf{i} + (t^4 - 2)\mathbf{j} + t^3\,\mathbf{k};$

$$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\,\mathbf{j} + 3t^3\,\mathbf{k};$$

$$\int \mathbf{F}\cdot d\mathbf{r} = \int_1^0 (5t^5 - 2t)\,dt = +\tfrac{1}{6}.$$

5. C_1 is defined in Exercise 3



(i) Line integral along C_2:

$x = t, \quad y = t, \quad z = t^2;$

$t_1 = 0, \quad t_2 = 1;$

$\mathbf{F} = 2t\,\mathbf{i} + (t^3 - 2)\mathbf{j} + t^2\,\mathbf{k};$

$$\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + 2t\,\mathbf{k};$$

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (2t + 3t^3 - 2)\,dt = -\tfrac{1}{4}.$$

(ii) For the curve C, we go along C_1 and return along C_2, so

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} - \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = -\tfrac{1}{6} - (-\tfrac{1}{4}) = \tfrac{1}{12}.$$

6. (i) If $\phi = \frac{x^3}{3} + \frac{y^2}{2}$

then $\operatorname{grad}\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$

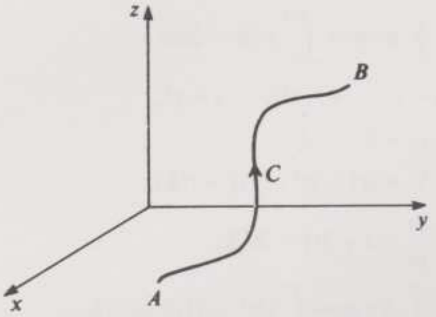
$$= x^2\,\mathbf{i} + y\mathbf{j}$$

and this is the vector field used in Example 4.

(ii) From Example 4,

$$\int_{AB} \mathbf{F}\cdot d\mathbf{r} = \left(\frac{x_2^3}{3} + \frac{y_2^2}{2}\right) - \left(\frac{x_1^3}{3} + \frac{y_1^2}{2}\right)$$
$$= \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1).$$

7. Suppose that A and B are any two points in the domain of a scalar field ϕ and let C be any curve joining A to B.



Let a parametrization of the curve be

$$x = f(t), \qquad y = g(t), \qquad z = h(t),$$

with $t = t_1$ and $t = t_2$ at A and B respectively.

Then $\operatorname{grad}\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$;

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k},$$

and hence

$$\operatorname{grad}\phi\cdot\frac{d\mathbf{r}}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}.$$

Using the chain rule from *Unit 25*, this can be written as $\frac{d\phi}{dt}$, thus

$$\int_{AB} \operatorname{grad}\phi\cdot d\mathbf{r} = \int_{t_1}^{t_2}\left(\operatorname{grad}\phi\cdot\frac{d\mathbf{r}}{dt}\right)dt$$
$$= \int_{t_1}^{t_2}\frac{d\phi}{dt}\,dt$$
$$= \phi(t_2) - \phi(t_1)$$
$$= \phi(B) - \phi(A).$$

This result is true for *all* paths joining A and B. Hence, the value of the line integral of $\operatorname{grad}\phi$ depends only on the end points of the path (i.e. A and B) and *not* on the path itself.

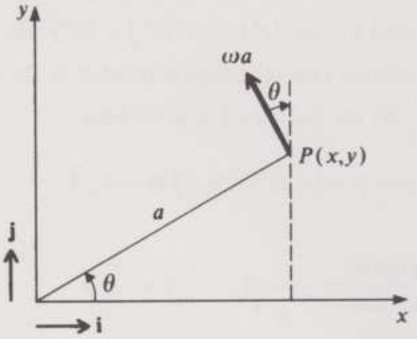
Solutions to the exercises in Section 4

1. (i) The velocity vector at the point P is given in terms of $\mathbf{i}$ and $\mathbf{j}$ by $\mathbf{u} = -\omega a\sin\theta\,\mathbf{i} + \omega a\cos\theta\,\mathbf{j}$

and $a\cos\theta = x$

$a\sin\theta = y$

so that $\mathbf{u} = -\omega y\mathbf{i} + \omega x\mathbf{j}$.

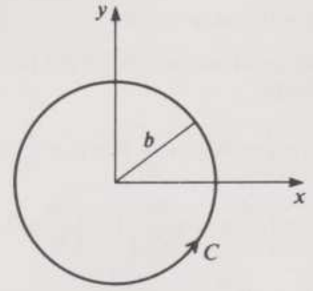


(ii) $x = b\cos t, \qquad y = b\sin t$;

$t_1 = 0,\ t_2 = 2\pi$;

$\mathbf{u} = -\omega b\sin t\,\mathbf{i} + \omega b\cos t\,\mathbf{j}$;

$\frac{d\mathbf{r}}{dt} = -b\sin t\,\mathbf{i} + b\cos t\,\mathbf{j}$.



$$\oint_C \mathbf{u}\cdot d\mathbf{r} = \int_0^{2\pi}(\omega b^2\sin^2 t + \omega b^2\cos^2 t)dt$$
$$= \int_0^{2\pi}\omega b^2 dt$$
$$= 2\pi\omega b^2.$$

2. $$f\left(y + \frac{\delta y}{2}\right) = f(y) + \frac{\delta y}{2}f'(y) + \left(\frac{\delta y}{2}\right)^2\frac{f''(y)}{2} + \left(\frac{\delta y}{2}\right)^3\frac{f'''(y)}{6} + \cdots$$

and $$f\left(y - \frac{\delta y}{2}\right) = f(y) - \frac{\delta y}{2}f'(y) + \left(\frac{\delta y}{2}\right)^2\frac{f''(y)}{2} - \left(\frac{\delta y}{2}\right)^3\frac{f'''(y)}{6} + \cdots .$$

Hence $$\frac{f\left(y - \frac{\delta y}{2}\right) - f\left(y + \frac{\delta y}{2}\right)}{\delta y} = -f'(y) - \frac{\delta y^2}{4}\frac{f'''(y)}{6} + \cdots$$

and $$\lim_{\delta y\to 0}\frac{f\left(y - \frac{\delta y}{2}\right) - f\left(y + \frac{\delta y}{2}\right)}{\delta y} = -f'(y) = -\frac{df}{dy}.$$

3. (i) For the square given in Figure 3,

$$\oint_C \mathbf{u}\cdot d\mathbf{r} = \int_{AB}\mathbf{u}\cdot d\mathbf{r} + \int_{BC}\mathbf{u}\cdot d\mathbf{r} + \int_{CD}\mathbf{u}\cdot d\mathbf{r} + \int_{DA}\mathbf{u}\cdot d\mathbf{r}.$$

Along AB, $\mathbf{r} = x\mathbf{i} - \frac{b}{2}\mathbf{j}$ since $y = -\frac{b}{2}$

so taking x as a parameter along here gives

$$\int_{AB}\mathbf{u}\cdot d\mathbf{r} = \int_{x=-b/2}^{x=b/2}\left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dx}\right)dx = \int_{-b/2}^{b/2}\frac{\omega b}{2}dx = \frac{\omega b^2}{2}.$$

Along BC, $\mathbf{r} = \frac{b}{2}\mathbf{i} + y\mathbf{j}$ since $x = \frac{b}{2}$ and y will be a parameter along here, giving

$$\int_{BC} \mathbf{u}\cdot d\mathbf{r} = \int_{-b/2}^{b/2} \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dy}\right) dy = \int_{-b/2}^{b/2} \frac{\omega b}{2}\, dy = \frac{\omega b^2}{2}.$$

Similarly $\int_{CD} \mathbf{u}\cdot d\mathbf{r} = \frac{\omega b^2}{2}$ and $\int_{DA} \mathbf{u}\cdot d\mathbf{r} = \frac{\omega b^2}{2}$.

Hence $\oint_C \mathbf{u}\cdot d\mathbf{r} = 2\omega b^2$. Now area $A = b^2$ for the square, so

$$\lim_{A\to 0} \frac{1}{A} \oint_C \mathbf{u}\cdot d\mathbf{r} = \lim_{A\to 0} \frac{1}{b^2}(2\omega b^2) = 2\omega.$$

(ii) $\oint_C \mathbf{u}\cdot d\mathbf{r} = \int_{AB} \mathbf{u}\cdot d\mathbf{r} + \int_{BC} \mathbf{u}\cdot d\mathbf{r} + \int_{CD} \mathbf{u}\cdot d\mathbf{r} + \int_{DA} \mathbf{u}\cdot d\mathbf{r}.$

Along AB, $\mathbf{r} = x\mathbf{i} + \left(y_0 - \frac{b}{2}\right)\mathbf{j}$, so

$$\int_{AB} \mathbf{u}\cdot d\mathbf{r} = \int_{x_0-b/2}^{x_0+b/2} -\omega\left(y_0 - \frac{b}{2}\right) dx$$
$$= -\omega b\left(y_0 - \frac{b}{2}\right).$$

Along BC, $\mathbf{r} = \left(x_0 + \frac{b}{2}\right)\mathbf{i} + y\mathbf{j}$, so

$$\int_{BC} \mathbf{u}\cdot d\mathbf{r} = \int_{y_0-b/2}^{y_0+b/2} \omega\left(x_0 + \frac{b}{2}\right) dy = \omega b\left(x_0 + \frac{b}{2}\right).$$

Along CD, $\mathbf{r} = x\mathbf{i} + \left(y_0 + \frac{b}{2}\right)\mathbf{j}$, so

$$\int_{CD} \mathbf{u}\cdot d\mathbf{r} = \int_{x_0+b/2}^{x_0-b/2} -\omega\left(y_0 + \frac{b}{2}\right) dx = \omega b\left(y_0 + \frac{b}{2}\right).$$

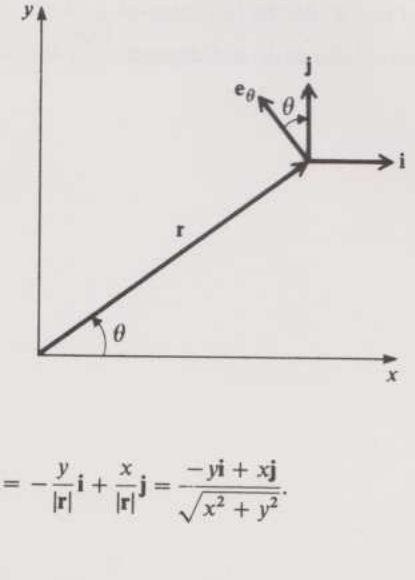
Along DA, $\mathbf{r} = \left(x_0 - \frac{b}{2}\right)\mathbf{i} + y\mathbf{j}$, so

$$\int_{DA} \mathbf{u}\cdot d\mathbf{r} = \int_{y_0+b/2}^{y_0-b/2} \omega\left(x_0 - \frac{b}{2}\right) dy = -\omega b\left(x_0 - \frac{b}{2}\right).$$

Thus $\oint_C \mathbf{u}\cdot d\mathbf{r} = 2\omega b^2$ so that $\lim_{A\to 0} \frac{1}{A} \oint_C \mathbf{u}\cdot d\mathbf{r} = 2\omega$ again.

4. **(i)** $\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$

and $\cos\theta = \frac{x}{|\mathbf{r}|}$ and $\sin\theta = \frac{y}{|\mathbf{r}|}$.

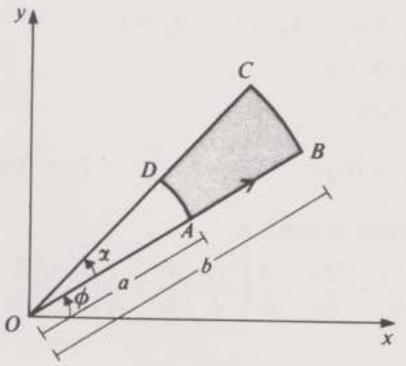


Hence $\mathbf{e}_\theta = -\frac{y}{|\mathbf{r}|}\mathbf{i} + \frac{x}{|\mathbf{r}|}\mathbf{j} = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}.$

(ii) $\mathbf{u} = \frac{c}{d}\mathbf{e}_\theta$, so using the result in part (i) we obtain

$$\mathbf{u} = \frac{c}{d}\left(-\frac{y}{d}\mathbf{i} + \frac{x}{d}\mathbf{j}\right) \qquad (\text{where } d = \sqrt{x^2 + y^2})$$
$$= \frac{c}{d^2}(-y\mathbf{i} + x\mathbf{j}).$$

(iii) Suppose that AD and BC are arcs of circles of radii a and b respectively.



$$\oint_C \mathbf{u}\cdot d\mathbf{r} = \int_{AB} \mathbf{u}\cdot d\mathbf{r} + \int_{BC} \mathbf{u}\cdot d\mathbf{r} + \int_{CD} \mathbf{u}\cdot d\mathbf{r} + \int_{DA} \mathbf{u}\cdot d\mathbf{r}$$

Along AB, $\mathbf{u}$ and $\frac{d\mathbf{r}}{dt}$ are perpendicular so that

$$\mathbf{u}\cdot\frac{d\mathbf{r}}{dt} = 0; \text{ so } \int_{AB} \mathbf{u}\cdot d\mathbf{r} = 0.$$

Similarly for the side CD, $\int_{CD} \mathbf{u}\cdot d\mathbf{r} = 0$.

Along BC, we can use the parametrization

$$x = b\cos t, \qquad y = b\sin t$$

and $t_1 = \phi$, $t_2 = \phi + \alpha$.

On substitution $\mathbf{u} = \frac{c}{b^2}(-b\sin t\,\mathbf{i} + b\cos t\,\mathbf{j})$,

$$\frac{d\mathbf{r}}{dt} = -b\sin t\,\mathbf{i} + b\cos t\,\mathbf{j},$$

so that $\mathbf{u}\cdot\frac{d\mathbf{r}}{dt} = c$ and $\int_{\phi}^{\phi+\alpha} \left(\mathbf{u}\cdot\frac{d\mathbf{r}}{dt}\right) dt = c\alpha.$

Similarly, along DA,

$$x = a\cos t, \qquad y = a\sin t$$

and $t_1 = \phi + \alpha$, $t_2 = \phi$; so

$$\int_{DA} \mathbf{u}\cdot d\mathbf{r} = \int_{\phi+\alpha}^{\phi} c\, dt = -c\alpha.$$

Hence $\oint_C \mathbf{u}\cdot d\mathbf{r} = 0 + c\alpha + 0 - c\alpha = 0.$

(iv) $(\operatorname{curl}\mathbf{u})_z = \lim_{A\to 0} \frac{1}{A} \oint_C \mathbf{u}\cdot d\mathbf{r} = 0$ since $\oint_C \mathbf{u}\cdot d\mathbf{r} = 0$.

5. $(\operatorname{curl}\mathbf{u})_z = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

(i) $\mathbf{u} = -\omega y\mathbf{i} + \omega x\mathbf{j}$

so $\frac{\partial F_2}{\partial x} = \omega$ and $\frac{\partial F_1}{\partial y} = -\omega$

so $(\operatorname{curl}\mathbf{u})_z = 2\omega$.

(ii) $\mathbf{u} = \dfrac{-cy}{(x^2+y^2)}\mathbf{i} + \dfrac{cx}{(x^2+y^2)}\mathbf{j}$ so

$$\frac{\partial F_2}{\partial x} = \frac{c}{(x^2+y^2)} - \frac{2cx^2}{(x^2+y^2)^2} = \frac{(y^2-x^2)c}{(x^2+y^2)^2},$$

$$\frac{\partial F_1}{\partial y} = \frac{-c}{(x^2+y^2)} + \frac{2cy^2}{(x^2+y^2)^2} = \frac{(y^2-x^2)c}{(x^2+y^2)^2}$$

so $(\operatorname{curl}\mathbf{u})_z = 0$.

6. **(i)** $\mathbf{F} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$;

$$F_1 = x, \qquad F_2 = y, \qquad F_3 = z;$$

$$\operatorname{curl}\mathbf{F} = \mathbf{0}.$$

(ii) $\mathbf{F} = f(r)\hat{\mathbf{r}}$;

$$F_1 = f(r)\frac{x}{r}, \qquad F_2 = f(r)\frac{y}{r}, \qquad F_3 = f(r)\frac{z}{r};$$

i component of curl**F** is

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \frac{d}{dr}\left(\frac{f(r)z}{r}\right)\frac{dr}{dy} - \frac{d}{dr}\left(\frac{f(r)y}{r}\right)\frac{dr}{dz}$$

(using the Chain Rule)

$$= z\frac{d}{dr}\left(\frac{f}{r}\right)\frac{y}{r} - y\frac{d}{dr}\left(\frac{f}{r}\right)\frac{z}{r} = 0.$$

Similarly the **j** and **k** components of curl**F** are zero, so curl $(f(r)\hat{\mathbf{r}}) = \mathbf{0}$ for all functions f of r.

(iii) $\mathbf{F} = (y^2+2z)\mathbf{i} + (xy+6z)\mathbf{j} + (z^2+2xz+y)\mathbf{k}$;

i component of curl**F** is $\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z} = 1 - 6 = -5$;

j component of curl**F** is $\dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x} = 2 - 2z = 2(1-z)$;

k component of curl**F** is $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = y - 2y = -y$;

so $\operatorname{curl}\mathbf{F} = -5\mathbf{i} + 2(1-z)\mathbf{j} - y\mathbf{k}$.

(iv) $\mathbf{F} = \operatorname{grad}\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$;

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right) = 0;$$

$$\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right) = 0;$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = 0;$$

hence curl (grad ϕ) = **0**.

7. $\nabla \times \mathbf{F} = \left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}\right) \times (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})$

and using the formula for the cross product given in *Unit 14* this becomes

$$\mathbf{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \operatorname{curl}\mathbf{F}.$$

8. **(i)** $\mathbf{F} = 2x\,\mathbf{i} + (xz-2)\mathbf{j} + xy\,\mathbf{k}$;

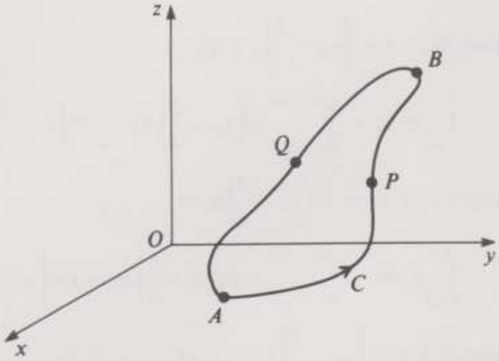
$$\operatorname{curl}\mathbf{F} = \mathbf{i}(x-x) + \mathbf{j}(0-y) + \mathbf{k}(z-0)$$

$$= -y\mathbf{j} + z\mathbf{k}.$$

Since $\operatorname{curl}\mathbf{F} \neq \mathbf{0}$ in general, **F** is non-conservative.

(ii) From Exercise 6(ii) $\operatorname{curl}(f(r)\hat{\mathbf{r}}) = \mathbf{0}$ for all functions $f(r)$. For $\mathbf{G} = r^n\hat{\mathbf{r}}$, $f(r) = r^n$ so that curl $\mathbf{G} = \mathbf{0}$ and hence **G** is conservative

9. If **F** is conservative then $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for all closed curves C. Consider a closed curve C passing through the two points A and B.



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{APB} \mathbf{F}\cdot d\mathbf{r} + \int_{BQA} \mathbf{F}\cdot d\mathbf{r} = 0.$$

Thus $\displaystyle\int_{APB} \mathbf{F}\cdot d\mathbf{r} = -\int_{BQA} \mathbf{F}\cdot d\mathbf{r}$.

But $\displaystyle\int_{BQA} \mathbf{F}\cdot d\mathbf{r} = -\int_{AQB} \mathbf{F}\cdot d\mathbf{r}$;

hence $\displaystyle\int_{APB} \mathbf{F}\cdot d\mathbf{r} = \int_{AQB} \mathbf{F}\cdot d\mathbf{r}$ and this is true for any path between A and B. So the line integral $\displaystyle\int_{AB} \mathbf{F}\cdot d\mathbf{r}$ is independent of the path, and depends only on the end points A and B.

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